Homework 1 Solutions

1. Below is the spectrum of the ^{13}CO $(1 \rightarrow 0)$ rotational line averaged over the entire Orion cloud.

The linewidth, σ_v, of a spectral line gives us a measure of the velocity dispersion in the line of sight. The linewidth can be thought of as the standard deviation (or 2nd moment) of the line:

$$\sigma_v^2 = \frac{\sum T(v)(v - v_0)^2}{\sum T(v)} km^2 s^{-2} \tag{1}$$

where we are summing over the channels of the receiver backend.

a. Use the Maxwellian distribution to calculate σ_v for a gas of pure molecular hydrogen at a temperature T_K.

$$f(v) = \left(\frac{1}{2\pi\mu m_H kT}\right)^{1/2} e^{-\frac{\mu m_H (v-v_0)^2}{2kT}} \tag{2}$$

This is a gaussian with a standard deviation of:

$$\sigma_v = \sqrt{\frac{kT}{\mu m_H}} = c_s \tag{3}$$

The value of μ is 2 for pure molecular hydrogen, or 2.7 for a standard mix of H_2 and He.

b. Given the answer of part a., what value does σ_v need to be for the linewidth to be in excess of the thermal sound speed, i.e. supersonic?

You could say that if $\sigma_v > \sqrt{\frac{kT}{\mu m_H}}$ that the gas is supersonic. However, since the gas contains a mixture of thermal and non-thermal motions, one must first subtract out the non-thermal motions.

$$v_{NT} = \sqrt{\sigma^2 - \sigma_{th}^2} \tag{4}$$

Where σ_{th} is the thermal velocity for the **observed molecule**. This is given by $\sigma_{th} = \sqrt{kT/\mu_{mol} m_H}$ where μ_{mol} is the atomic weight of the relevant molecule (for example, $\mu_{mol} = 28$ for CO). The non-thermal motions are supersonic if $v_{NT}/c_s > 1$ or:

$$\frac{\sqrt{\sigma^2 - \sigma_{th}^2}}{c_s} > 1 \tag{5}$$

Remember that c_s depends on the mean molecular weight for the gas (there is just one sound speed for the gas, not an individual sound speed for each species).

c. A typical spectral resolution for a millimeter wave receiver is $\Delta f = 50$ MHz. For observations of ^{13}CO $(1 \rightarrow 0)$ at 110.2 GHz, what is the resolution in velocity, Δv? Can you resolve purely thermal motions in a 20 K gas?

In reality, the resolution of a spectrometer is $\Delta f = 50$ kHz. Oops. In any case, we use the equation for the Doppler shift.

$$\frac{\Delta f}{f} = \frac{v}{c} \tag{6}$$

so that a resolution of $\Delta f = 50$ MHz at 110.2 GHz is 136 km s^{-1} and $\Delta f = 50$ kHz is 0.14 km s^{-1}. At a resolution of 50 MHz, one could not observe thermal motions, but with 50 kHz, one could.

2. An interesting implication of the Jeans instability was first described by Fred Hoyle in 1953. Imagine a clump of gas with $M = M_J = 100$ $M_\odot$ becomes unstable and collapses (M_J is the Jeans mass). What happens to the Jeans mass of the clump as it collapses? Describe what can happen as the density increases by a factor of 4, 16, 64 , ... and so on. Do you ever reach the point where $M_J \sim 1$ $M_\odot$?

The Jeans mass is given by.

$$M_J = \left(\frac{\pi k T}{\mu m_H G}\right)^{3/2} \rho_0^{-1/2} \tag{7}$$

Thus, if the density increases by 4, the Jeans mass is 50 M$_\odot$, by 16 it is 25 M$_\odot$, and 64 it is 12.5 M$_\odot$. To go to 1 M$_\odot$, the density must be increased by a factor of 10,000.

3. Shocks in a supersonically turbulent gas can affect the Jeans mass. Imagine the collision of two clumps of gas moving relative to each other supersonic velocities.How fast must the shock be to obtain Jeans masses of 1 $M_\odot$ in an initially 1000 cm^{-3}, $T_K = 20$ K gas? Is this at all consistent with the linewidth in problem 1? To convert between the mass density and number density, use $\mu = 2.7$.

The post-shock gas has a density of

$$\rho = \rho_0 \left(\frac{v_s}{c_s}\right)^2 = \rho_0 v_s^2 \frac{\mu m_h}{kT} \tag{8}$$

Given the equation of a Jeans mass in Question 2, $\rho = 1 \times 10^6\ cm^{-3}$ for $M_J = 1\ \mathrm{M}_\odot$ at 20 K.

$$\rho = \rho_0 \left(\frac{v_s}{c_s}\right)^2 = \rho_0 v_s^2 \frac{\mu m_H}{kT} \tag{9}$$

Solving for v_s we find

$$v_s = \sqrt{\frac{\rho}{\rho_0}\left(\frac{kT}{\mu m_H}\right)} = \sqrt{\frac{10^6\ cm^{-3}}{10^3\ cm^{-3}}} \cdot 0.25\ km\ s^{-1} = 8\ km\ s^{-1} \tag{10}$$