Homework 3 Solutions

1. Consider a constant velocity flow where there is no acceleration (either infall or outflow, the direction of the flow is irrelevant). The density is given by:

$$\rho \propto r^{-\alpha} \tag{1}$$

Using the equation of continuity, what is α?

The continuity equation in spherical coordinates is given by:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \rho v = 0 \tag{2}$$

Given that the density is stationary, i.e. it does not change with time, then:

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \rho v = 0 \tag{3}$$

this can only happen if $r^2\rho v = constant$. Since v is also constant, then:

$$\rho \propto \frac{1}{r^2} \tag{4}$$

2. What is the time dependence for the this initial phase of a Bonnor-Ebert sphere collapse?

We assume that the density of gas is constant with a value of ρ. The collapse of the gas in initiated by the passage of the rarefaction wave that travels with speed c_s. Thus, at time t, the material at radius $c_s t$ collapse. Since the gas has a constant density, the free fall time of any shell is the same. Thus, the infall rate can be given by speed of the rarefaction wave, times the density of the material, times the volume of the sphere which is being triggered: $dV = 4\pi r^2 dr = 4\pi c_s^3 t^2$:

$$\dot{M} = 4\pi \rho c_s^3 t^2 \tag{5}$$

(Note, that the actual time for the material to arrive at the protostar will be delayed by one free fall time.) This shows why there is a large ramp up in the infall rate in the early stages of the collapse of a Bonner-Ebert sphere.

2. Derive the spectral slope of a flat disk. Make the following assumptions

$$T = T_0 \left(\frac{r}{r_0}\right)^{-q} \tag{6}$$

where T_0, r_0 and q are constants (T_0 is the temperature at r_0).

i.) Use the equation for the flux density from a disk inclined at an angle θ.

$$\nu F_\nu = \frac{2\pi cos(\theta)\nu}{D^2} \int_{R_{in}}^{R_{out}} B_\nu(T) r dr \tag{7}$$

we now use the equation for a Planck function

$$\nu F_\nu = \frac{2\pi cos(\theta)\nu}{D^2} \int_{R_{in}}^{R_{out}} \frac{2hv^3}{c^2} \frac{1}{e^{\left(\frac{h\nu}{kT_0}\right)\left(\frac{r}{r_0}\right)^q} - 1} r dr \tag{8}$$

Now we switch variables

$$x \equiv \left(\frac{h\nu}{kT_0}\right)^{1/q} \frac{r}{r_0},\ r = \left(\frac{kT_0}{h\nu}\right)^{1/q} r_0 x \tag{9}$$

The result is

$$\nu F_\nu = \frac{2\pi cos(\theta) r_0^2 \nu}{D^2} \left(\frac{2hv^3}{c^2}\right) \left(\frac{kT_0}{h\nu}\right)^{2/q} \int_{X_{in}}^{X_{out}} \frac{1}{e^{x^q} - 1} x dx \tag{10}$$

We let $X_{in} \rightarrow 0$ and $X_{out} \rightarrow \infty$. Although this will mean the short and long wavelength regions of the SED will be incorrect, this is good for the mid-IR portion of the SED. We are also assuming that the disk is optically thick at all wavelengths; however, at longer wavelengths, this approximation will break down. The resulting dependence on frequency is then:

$$\nu F_\nu \propto \nu^{4-2/q} \tag{11}$$

ii.) What is the spectral index:

$$\alpha = \frac{dlog(\nu F_\nu)}{dlog(\nu)} \tag{12}$$

as a function of q? What is α for a passively heated flat disk where $q = 3/4$?

$$\alpha = 4 - 2/q \tag{13}$$

which implies that $\alpha = 4/3$ when $q = 3/4$. Note, if we measure α in terms of wavelength ($\alpha = dlog(\lambda F_\lambda)dlog(\lambda)$), then the sign switches. Thus, when we classify YSOs with α, be careful if it is measured as a slope relative to $log(\nu)$ or $\log(\lambda)$. The classification of YSOs depends on the slope relative to $\log(\lambda)$, at which the the slope of a flat disk will be $\alpha = -4/3$.

3. Write down the radiative transfer equation of a two layer disk, the two layers being a dense inner layer centered on the mid-plane of the disk and the outer layer being a thinner disk atmosphere. Consider each layer as an infinite slab of constant thickness. Consider only radiative transfer perpendicular to the slab (reducing this to a 1-D problem). First, write the equation for the intensity of light emitted perpendicular to the slabs in terms of the temperature of the inner slab T_2, outer slab temperature T_1, the opacity per mass κ_ν, the density of the outer slab ρ_1, and the thickness of the outer slab, H_1. Assume that the inner slab has $\tau_2 >> 1$.

$$I_\nu = B_\nu(T_2)e^{-\tau_1} + B_\nu(T_1)(1 - e^{-\tau_1}) \tag{14}$$

where $\tau_1 = \kappa_\nu \rho H_1$.

In which of the following cases do you see the silicate features in emission, in absorption, or no silicate feature?

Case 1: $T_1 > T_2,\ \tau_1 << 1$; Case 2: $T_1 > T_2,\ \tau_1 >> 1$; Case 3: $T_1 < T_2,\ \tau_1 << 1$

For Case 1, we get the silicate features in emission (higher temperatures and low optical depth). For Case 2, we get no emission features (the outer layer is optically thick so we just get a blackbody of temperature T_1). For Case 3, we get the silicate features in absorption (cooler temperatures and low optical depth).