

Homework 4 Solutions

1. *Fun physics with mean molecular/atomic weight (μ).*

Why does the pitch of your voice change if you inhale helium? Assume the size of your vocal cords stay the same (this sets the wavelength of the sound waves). How does the pitch change when you go from a Nitrogen/Oxygen air to pure Helium, i.e. what is the ratio of sound speeds in a normal and Helium atmosphere, and what is the resulting ratio in frequencies?

We can write μ as:

$$\frac{1}{\mu} = \sum \frac{X_i}{A_i} \quad (1)$$

where X_i is the mass fraction of species i and A_i is the total atomic weight of the species. We are assuming in this case neutral species with no free electrons. For molecules, the atomic weight is the sum of the atomic weight of the constituent atoms. For pure, neutral Helium, $X = 1$ and $A = 4$.

$$\frac{1}{\mu} = \frac{1}{4} \quad (2)$$

For air, let's assume a N_2 and O_2 mixture. For N_2 the total atomic weight is $A = 28$ and for O_2 the total weight is $A = 32$. The molar fraction are 78% and 21%. How does this correspond to the mass fraction? Let's begin by writing the number densities of N_2 and O_2 :

$$n(N_2) = \frac{\rho X(N_2)}{28m_H}, \quad n(O_2) = \frac{\rho X(O_2)}{32m_H} \quad (3)$$

Also, the total number of particles

$$n_{tot} = \frac{\rho}{\mu m_H} \quad (4)$$

Thus, the molar fractions is given by

$$f(N_2) = \frac{n(N_2)}{n_{tot}} = X(N_2) \frac{\mu}{28}, \quad f(O_2) = \frac{n(O_2)}{n_{tot}} = X(O_2) \frac{\mu}{32} \quad (5)$$

Since $X(N_2) + X(O_2) \approx 1$ we can ignore the other constituents of air:

$$\mu = 32f(O_2) + 28f(N_2) = 28.88 \quad (6)$$

Note, if we take into account the next most abundant gas, Argon, it goes from 28.88 to 29.28. I will adopt 29. Now, assuming an isothermal gas, the sound speed is:

$$c_s = \sqrt{\frac{kT}{\mu m_H}} \quad (7)$$

Since the wavelength is set by the length of the vocal cord, and the pitch is equal to $f = c_s/\lambda$, then the ratio of the pitches is equal to the ratio of the sound speeds:

$$\frac{f(He)}{f(Air)} = \frac{c_s(He)}{c_s(Air)} = \sqrt{\frac{\mu(air)}{\mu(He)}} = 2.7 \quad (8)$$

b. *Would a pure helium balloon float in the atmosphere of Jupiter? The atmosphere of Jupiter is 90% molecular Hydrogen and 10% Helium (by number, not weight). Calculate the net force on the balloon (you will have to look up the mass and radius of Jupiter) as a function of the surrounding atmospheric pressure and temperature. Assume the temperature in the balloon is the same as that in the surrounding atmosphere. This is not hard, you just need a Eureka!! moment.*

We apply here Archimedes principle, which underlies the idea of buoyancy. Imagine the atmosphere in hydrostatic equilibrium:

$$\frac{dP}{dz} = -\rho_{Jov}g \quad (9)$$

where z is altitude. Now, we substitute a balloon in the atmosphere. The net force on the balloon is the pressure minus the weight (F_z is the force in the upward direction, which requires a negative gradient in pressure so the force from the atmospheric pressure will be larger at the bottom of the ballon than the top). For simplicity, assume a cylindrical balloon with an area A at the top and bottom and a height H :

$$F_z = -\frac{dP}{dz}HA - HA\rho_{bn}g = HA g(\rho_{Jov} - \rho_n) \quad (10)$$

were ρ_{bn} is the density of the gas. We have assumed that the weight of walls of the ballon is negligible. We can calculate the densities of both the Jovian atmosphere and the balloon since they are in pressure and temperature equilibrium. Using the ideal gas law, $\rho = \mu m_H P/kT$

$$F_z = H A g m_H \frac{P}{kT} (\mu_{Jov} - \mu_{bn}) \quad (11)$$

The values of μ are $\mu_{Jov} = 0.9 \times 2 + 0.1 \times 4 = 2.2$ and $\mu_{bn} = 4$ (pure He). I have assumed that the Hydrogen is in the form of molecular Hydrogen. Since the $\mu_{bn} > \mu_{Jov}$, the resulting force will be negative, and the balloon will sink.

2. *Determine the equation for the luminosity evolution of a main sequence star. The equation should be in terms of the initial luminosity and m at $t=0$, the mass of the star, and the energy generation per gram of gas. First calculate the rate at which Hydrogen is converted into Helium and from this get dX/dt (assume the star is just Helium and Hydrogen). From this measure the time variation of μ . Then use the variation of μ with time to get the change in luminosity.*

In Lecture 21 equations we found:

$$\mu \approx \frac{4}{3 + 5X} \quad (12)$$

and

$$L \propto \frac{M^{5.53} \rho^{0.166} \mu^{7.5}}{\kappa_0} \quad (13)$$

which can be written as:

$$\frac{L(t)}{L(0)} = \left[\frac{\mu(t)}{\mu(0)} \right]^{7.5} \quad (14)$$

First, let's calculate the time dependence of μ . We first write:

$$\frac{d\mu}{dt} = -\frac{20}{(3 + 5X)^2} \frac{dX}{dt} = -\frac{5}{4} \mu^2 \frac{L}{\epsilon M} \quad (15)$$

Here we use the rate of nuclear fusion to find determine the depletion of Hydrogen, i.e. $dX/dT = L/\epsilon M$ where L is the luminosity of the star, ϵ is the nuclear energy produced per gram, M is the mass of the star. Now we substitute the equation for the luminosity:

$$\frac{d\mu(t)}{dt} = -\frac{5}{4} \mu(t)^2 \frac{L(0)}{\epsilon M} \left[\frac{\mu(t)}{\mu(0)} \right]^{7.5} = -\frac{5}{4} \left[\frac{\mu(t)^{9.5}}{\mu(0)^{7.5}} \right] \frac{L(0)}{\epsilon M} \quad (16)$$

Defining

$$K_1 = \frac{5 L(0)}{4 \epsilon M} \quad (17)$$

Now we solve this equation by:

$$dT = -\frac{\mu(0)^{7.5}}{K_1} \mu^{-9.5} d\mu \quad (18)$$

using the initial condition that $\mu(t) = \mu(0)$ at $t = 0$, we integrate the above equation to get.

$$t = \frac{\mu(0)^{7.5}}{8.5 K_1} (\mu^{-8.5} - \mu(0)^{-8.5}) \quad (19)$$

We then invert the equation to get:

$$\mu(t) = \mu(0) \left(1 - \frac{17}{2} K_1 \mu(0) t \right)^{-2/17} \quad (20)$$

Finally, we plug this back into the luminosity:

$$L(t) = L(0) \left(1 - \frac{17}{2} K_1 \mu(0) t \right)^{-15/17} \quad (21)$$

Substituting back into K_1

$$\mu(t) = \mu(0) \left(1 - \frac{85 \mu(0) L(0)}{8 \epsilon M} t \right)^{-2/17} \quad (22)$$

$$L(t) = L(0) \left(1 - \frac{85 \mu(0) L(0)}{8 \epsilon M} t \right)^{-15/17} \quad (23)$$

3. *What mass infall rate is needed to sustain constant Deuterium burning in the center of a protostar? Assume a standard abundance and use a protostar mass of $0.5 M_\odot$ and a radius of $2 R_\odot$. Also assume the protostar is convective so that Deuterium landing on the surface of the star is transported to the center of the star.*

From lecture 18, we can write the change in the mass in Deuterium as:

$$\frac{dfM}{dt} = \dot{M} - \frac{L_D}{\beta_D} \quad (24)$$

where \dot{M} is the mass accretion, dfM/dt is the change in the mass of Deuterium with time (f is the current fraction of mass in Deuterium over the fraction in mass in Deuterium assuming interstellar abundances), L_D is the Deuterium luminosity and β_D is the energy per gram of gas (assuming interstellar abundances). For the mass of Deuterium not to decrease, $dfM/dt = 0$ or

$$\dot{M} = \frac{L_D}{\beta_D} \quad (25)$$

The Deuterium luminosity is given by (Lecture 18):

$$L_D = 1.92 \times 10^{17} f \left[\frac{D}{H} \right] \left(\frac{M}{M_\odot} \right)^{13.8} \left(\frac{R}{R_\odot} \right)^{-14.8} L_\odot \quad (26)$$

which for $M = 0.5 M_\odot$, $R = 2 R_\odot$, $f = 1$, and $D/H = 2.5 \times 10^{-5}$ gives a luminosity of 11,000 L_\odot . This is very high. A more realistic radius, say $3 R_\odot$ would have resulted in a more realistic luminosity. Nevertheless, we can carry out the calculation.

$$\dot{M} = \frac{11,000 L_\odot \cdot 4 \times 10^{33} \text{ erg s}^{-1} L_\odot^{-1} \cdot \pi \times 10^7 \text{ s yr}^{-1}}{9.2 \times 10^{13} \text{ ergs gm}^{-1} \cdot 2 \times 10^{33} \text{ gm } M_\odot^{-1}} = 0.008 M_\odot \text{ yr}^{-1} \quad (27)$$

This is also very high, but a radius of $2.7 R_\odot$ and a luminosity around $100 L_\odot$ would have produced a more reasonable \dot{M} of $8 \times 10^{-5} M_\odot \text{ yr}^{-1}$.