Homework 5 Solutions

1. *Imagine two nuclei with charges* Z_1 *and* Z_2 *colliding. Show that if the particles are traveling at relative velocity* v *(where* $v = v_1 - v_2$*), the distance of closest approach (before the particle is repelled by the Coloumb barrier) is given by:*

$$r = \frac{2Z_1Z_2e^2}{mv^2} \tag{1}$$

As we discussed in class, this semester and last, the probability that the particle will tunnel through the Coulomb barrier is:

$$P \propto exp^{-4\pi^2 Z_1Z_2e^2/hv} \tag{2}$$

For a gas in thermodynamic equilibrium, the probability of having a velocity v *is given by the Maxwellian:*

$$P \propto exp^{-mv^2/2kT} \tag{3}$$

Multiply the two probabilities to get the probability of a thermonuclear reaction. Show that the maximum probability happens for a velocity of:

$$v = (4\pi^2 Z_1Z_2e^2kT/hm)^{1/3} \tag{4}$$

The multiplication of the two probabilities gives:

$$P \propto exp^{-4\pi^2 Z_1Z_2e^2/hv - mv^2/2kT} \tag{5}$$

The maximum probability can be found by minimizing the exponent. This minimum can be found by setting the derivative to zero:

$$\frac{d(-4\pi^2Z_1Z_2e^2/hv - mv^2/2kT)}{dt} = 4\pi^2Z_1Z_2e^2/hv^2 - mv/2kT = 0 \tag{6}$$

The solution is:

$$v = (4\pi^2 Z_1Z_2e^2kT/hm)^{1/3} \tag{7}$$

,

Multiply the two probabilities to get the probability of a thermonuclear reaction. Show that the maximum probability happens for a velocity of:

$$v = (4\pi^2 Z_1 Z_2 e^2 kT/hm)^{1/3} \tag{8}$$

For the first reaction of the PPI chain at a $T = 1.5 \times 10^7 K$*, calculate the closest approach* r *for this value of* v*.*

The radius will be

$$r = \frac{2e^2}{m_p}\left(\frac{4\pi^2 e^2 kT}{hm_p}\right)^{-2/3} = 20 \text{ fm} \tag{9}$$

where m_p is the mass of the proton and 1 fm $= 10^{-15}$ m.

Next show that the maximum probability will be

$$P_{max} \propto e^{-3/2(4\pi^2 Z_1 Z_2 e^2/h)^{2/3}(m/kT)^{1/3}} \tag{10}$$

If we plug the velocity back into the equation for the probability of a thermonuclear reaction, we get:

$$P \propto exp^{-(4\pi^2 Z_1 Z_2 e^2/h)^{2/3}(m/kT)^{1/3} - (4\pi^2 Z_1 Z_2 e^2 kT)(1/2)(m/kT)^{1/3}} \tag{11}$$

$$P \propto exp^{-(3/2)(4\pi^2 Z_1 Z_2 e^2/h)^{2/3}(m/kT)^{1/3}} \tag{12}$$

Finally, defining $T_0 = (3/2)^3(4\pi^2 Z_1 Z_2 e^2/h)^2(m/k)$*, plot the resulting probability*

$$P_{max} \propto e^{-(T_0/T)^{1/3}} \tag{13}$$

What does this probability say about the temperature sensitivity of the reaction?

Setting $Z_1 = 1$ and $Z_2 = 1$ and using the mass of a proton.

$$T_0 = 4 \times 10^7 K \tag{14}$$

The plot is on the next page.

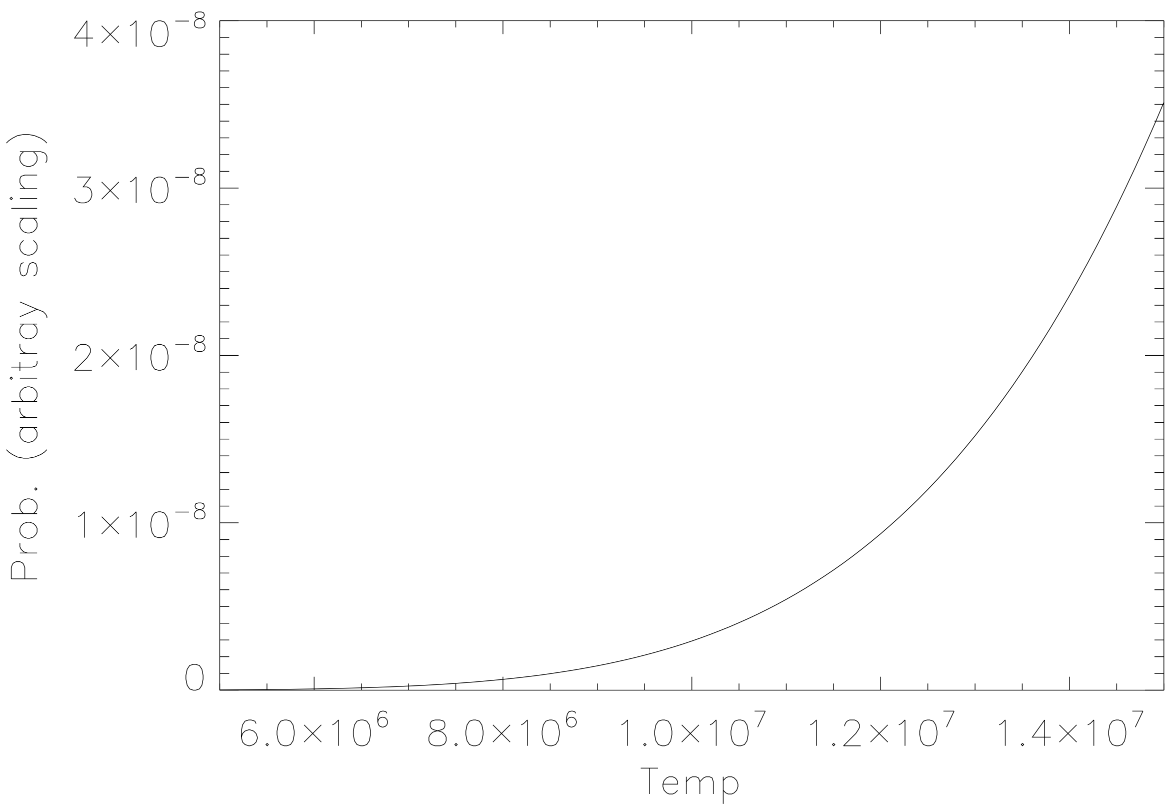

Fig. 1.— The plot of $P_{max} \propto e^{-(T_0/T)^{1/3}}$ as a function of T. The Y-axis scaling is arbitrary. This illustrates the very strong dependence of the rate of fusion on temperature.

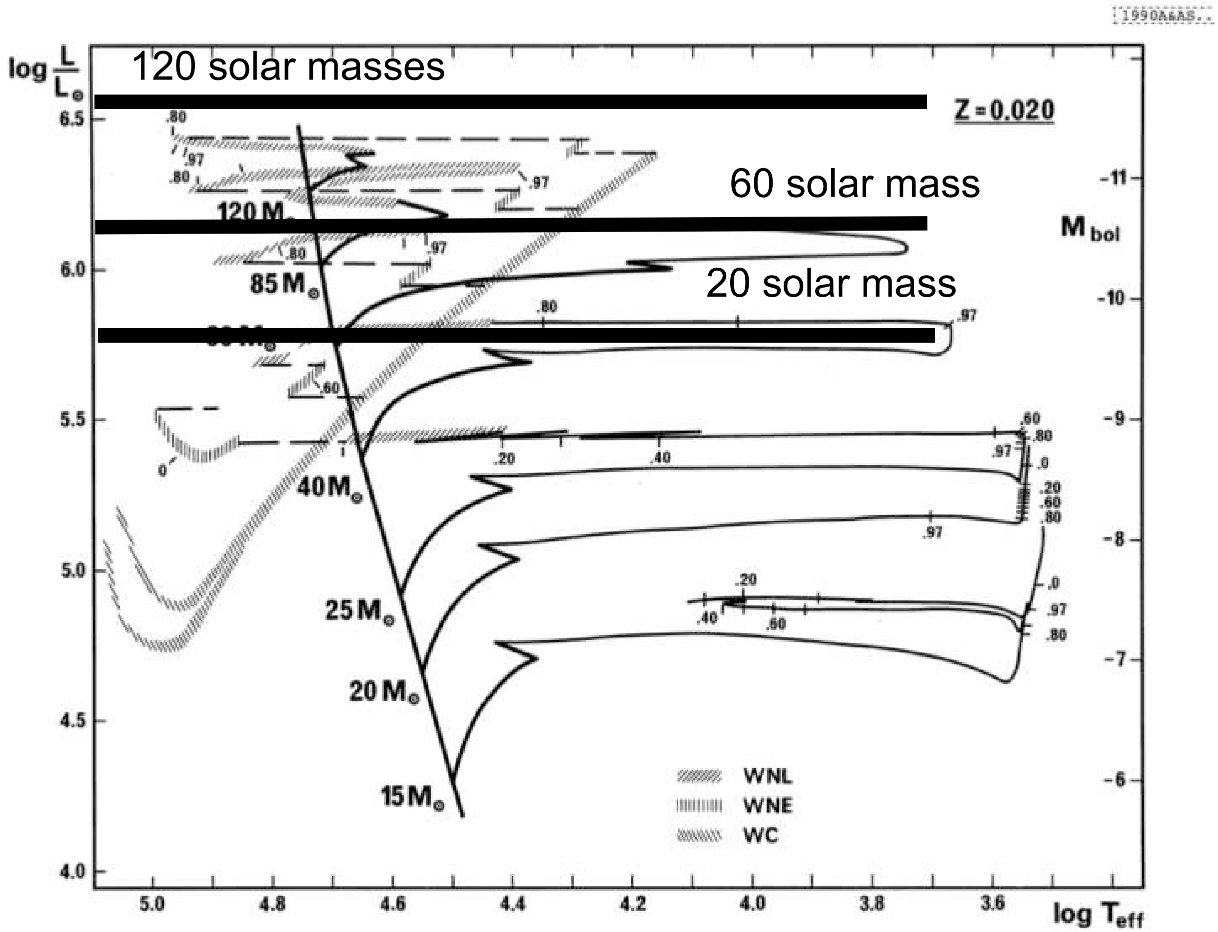

Fig. 2.— The lines show the Eddington limit for the labeled masses. Note that the 120 and 60 $M_\odot$ stars are close to their Eddington limits, particularly as they evolve off the main sequence.