1. *The Scale Height of a Disks*

Consider the force on a particle orbiting in a disk which is sitting a height z above a disk at a radius R. From the perspective of the central star, the angle between the particle and disk (with the central star at the apex) is given $\alpha \approx z/R$ (assuming $z << R$). The force can be considered two components are in cylindrical coordinates:

$$F = -[sin(\alpha)\hat{\mathbf{z}} + cos(\alpha)\hat{\mathbf{r}}]\frac{GmM}{R^2} \tag{1}$$

where M is the mass of the central star. In the case of $z << R$, this becomes:

$$F = -\left(\frac{z}{R}\hat{\mathbf{z}} + \hat{\mathbf{r}}\right)\frac{GmM}{R^2} \tag{2}$$

where $sin(\alpha) = z/R$. The radial component is balanced by centrifugal force due to the Keplerian rotation of the disk. In the vertical direction, we assume that the disk is locally supported by pressure, and it could be considered to be in hydrostatic equilibrium. In this case:

$$\frac{dP}{dz} = -\frac{z}{R}\frac{G\rho M}{R^2} \tag{3}$$

where z is now the distance of above the midplane. we have replaced the mass of the particle m by the density of the gas ρ. Assuming isothermal gas ($P = c_s^2\rho$):

$$\frac{d\rho}{dz} = -\frac{G\rho z M}{c_s^2 R^3} \tag{4}$$

The solution of this equation is:

$$\rho(z) = \rho(0)e^{-(GMz^2)/(2c_s^2R^2)} = \rho(0)e^{-\frac{z^2}{2H^2}} \tag{5}$$

where the scale height is

$$H = \left(\frac{R^3c_s^2}{GM}\right)^{\frac{1}{2}} \tag{6}$$

we can also write the scale height in terms of the Keplerian rotation speed $v_\phi = \sqrt{GM/R}$.

$$\frac{H}{R} = \frac{c_s}{v_\phi}, \text{ or } H = \frac{c_s}{\omega} \tag{7}$$

where $v_\phi = \omega R$,

2. *Emission from a Disk Surface*

Consider the region $R >> R_\star$ where a flat disk which show be irradiated by a very weak radiation field. Now consider a small square on a flared disk surface titled at angle α relative to the line of sight to the star. That angle α is related to the scale height by:

$$sin(\alpha) = \frac{dH}{dR} - \frac{H}{R} \tag{8}$$

where H is the height of the disk surface (i.e photosphere) at radius R. We can also write $sin(\alpha)$ as:

$$sin(\alpha) = R\frac{\partial}{\partial R}\left(\frac{H}{R}\right) \tag{9}$$

$$F_{irr} = sin(\alpha)\frac{L_\star}{4\pi R^2} \tag{10}$$

the cooling flux is

$$F_{cool} = \sigma T^4 \tag{11}$$

Equating heating and cooling:

$$T^4 = sin(\alpha)\frac{L_\star}{\sigma 4\pi R^2} \tag{12}$$

If we write

$$sin(\alpha) = \xi\frac{H}{R} \tag{13}$$

$$T^4 = \frac{\xi H L_\star}{4\pi\sigma R^3} \tag{14}$$

Now let's write the surface H as a function of the pressure scale height, $H = \chi h$. Then

$$T^4 = \frac{\xi \chi h L_\star}{4\pi\sigma R^3} \tag{15}$$

where

$$h = \sqrt{\frac{kTR^3}{\mu m_H G M_\star}},\ h^8 = \left(\frac{kTR^3}{\mu m_H G M_\star}\right)^4 r^{12} T^4 \tag{16}$$

substituting equation

$$h^7 = \left(\frac{kTR^3}{\mu m_H G M_\star}\right)^4 \frac{\xi \chi h L_\star}{4\pi\sigma} r^9 \tag{17}$$

or

$$h \propto r^{9/7}, T \propto r^{-3/7} \tag{18}$$

Thus, the temperature gradient is much less steep than $T \propto r^{-3/4}$ for flat disks. Note that for a flared disk:

$$T^4 = sin(\alpha) \frac{L_\star}{4\pi\sigma R^2} \tag{19}$$

while for an individual greybody dust grain:

$$T^4 = \frac{L_\star}{16\pi\sigma R^2} \tag{20}$$