

1. Mass of Disk (from Hartmann)

Let us assume that the temperature and surface density follow a power law:

$$T = T_0 \left(\frac{R}{R_0} \right)^{-q} \quad (1)$$

$$\Sigma = \Sigma_0 \left(\frac{R}{R_0} \right)^{-p} \quad (2)$$

Then if $\tau_\nu \ll 1$ then, where $\tau_\nu = \kappa_\nu \Sigma$ where κ_ν is the absorption per mass

$$\nu L_\nu = 4\pi \int_{R_i}^{R_d} \nu B_\nu \kappa_\nu \Sigma(R) 2\pi R dR \quad (3)$$

$$\nu L_\nu = \frac{16\pi^2 \kappa}{c^2} \nu^3 \kappa_\nu \Sigma_0 T_0 R_0^2 \frac{(R_d/R_0)^{2-p-q}}{(2-p-q)} \quad (4)$$

and

$$M_d = \int_{R_i}^{R_d} 2\pi \Sigma R dR = 2\pi \Sigma_0 R_0^2 \frac{(R_d/R_0)^{2-p}}{(2-p)} \quad (5)$$

which substituting for Σ

$$\nu L_\nu = \frac{8\pi k \nu^3 \kappa_\nu}{c^2} M_d T_0 \left(\frac{R_d}{R_0} \right)^{-q} \frac{2-p}{2-p-q} \quad (6)$$

2. Size of Dust Grains and Opacity

Imagine you have a constant mass of dust in a disk, but the size of grains are growing. Let's assume that all the grains have the same radius, a , and the same density ρ . Then the number of grains is simply:

$$N = \frac{3M_{dust}}{4\pi\rho a^3} \quad (7)$$

The geometric cross section of those same grains will be

$$\sigma_{geo} = \frac{3\pi a^2 M_{dust}}{4\pi\rho a^3} = \frac{3M_{dust}}{4\rho a} \quad (8)$$

Thus the cross section of the grains decrease with size. In other words, as the grains collide and coagulate into larger bodies, the cross-section drops. As grains go from $1 \mu\text{m}$ to 1m , the cross section drops by 10^{-6} . Consider an optically thin disk made out of grains with size a . The thermal emission is:

$$L_\nu = 4\sigma_{geo}\pi B_\nu(T) = \frac{3M_{dust}\pi B_\nu(T)}{\rho a} \quad (9)$$

Thus the luminosity from the dust grains drops by 10^{-6} as grains go from $1 \mu\text{m}$ to 1m in size.

3. Poynting Robertson Drag

The Poynting Robertson drag is on a grain of mass m_d and radius a orbiting with a velocity v_ϕ at an orbital radius r from a star with luminosity L_\star is given by:

$$F_{PR} = \frac{L_\star}{4\pi r^2 c} \frac{3\pi a^2 m_d v_\phi}{4\pi\rho a^3} \frac{1}{c} \quad (10)$$

where the first term is the momentum flux carried by the photons from a star, the 2nd term is cross section of the dust, the last term is the fraction of the momentum flux intercepted in the direction of motion. We can also write this as a torque.

$$\frac{dm_d r v_\phi}{dt} = \frac{L_\star}{4\pi r^2 c} \frac{3\pi a^2 m_d r v_\phi}{4\pi a^3} \frac{1}{c} \quad (11)$$

$$\frac{drv_\phi}{dt} = \frac{L_\star}{4\pi r^2 c} \left(\frac{3\pi a^2}{4\pi \rho a^3} \right) \frac{rv_\phi}{c} \quad (12)$$

By substituting keplerian rotation $rv_\phi = (GM_\star r)^{1/2}$ and assuming circular orbits, we get the equation:

$$\frac{dr^{1/2}}{dt} = \frac{L_\star}{4\pi r^{3/2} c^2} \left(\frac{3}{4a} \right) \quad (13)$$

Thus, $t_{PR} \approx r^{1/2}/(dr^{1/2}/dt)$ is given by

$$t_{PR} \approx \frac{4\pi a \rho}{3} \left(\frac{c^2 r^2}{L_\star} \right) \quad (14)$$

or

$$t_{PR} \approx 2.2 \times 10^3 \left(\frac{a}{1\mu m} \right) \left(\frac{\rho}{3 \text{ g cm}^{-3}} \right) \left(\frac{r^2}{1AU^2} \right) \left(\frac{L_\star}{L_\odot} \right) yr \quad (15)$$