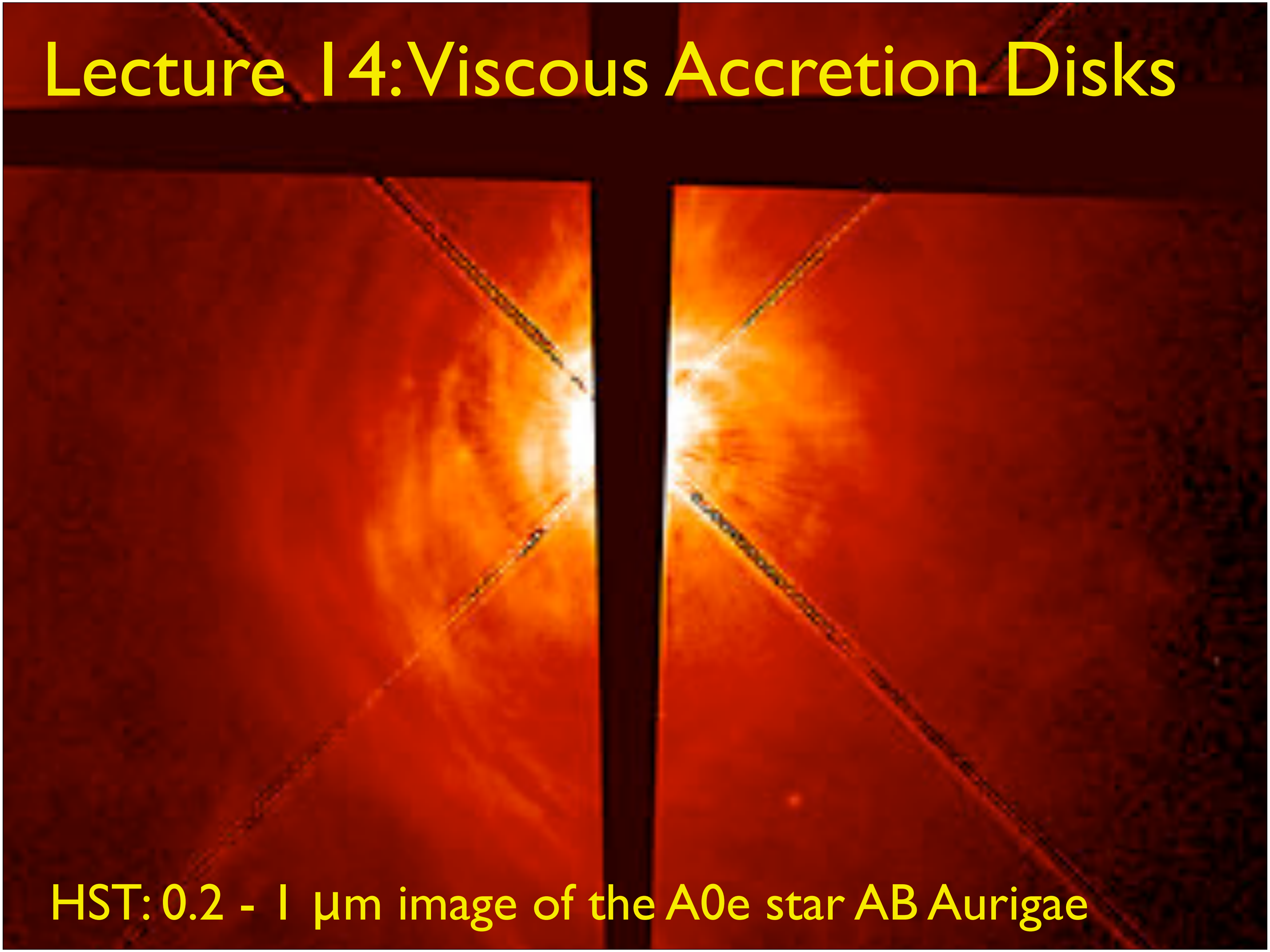
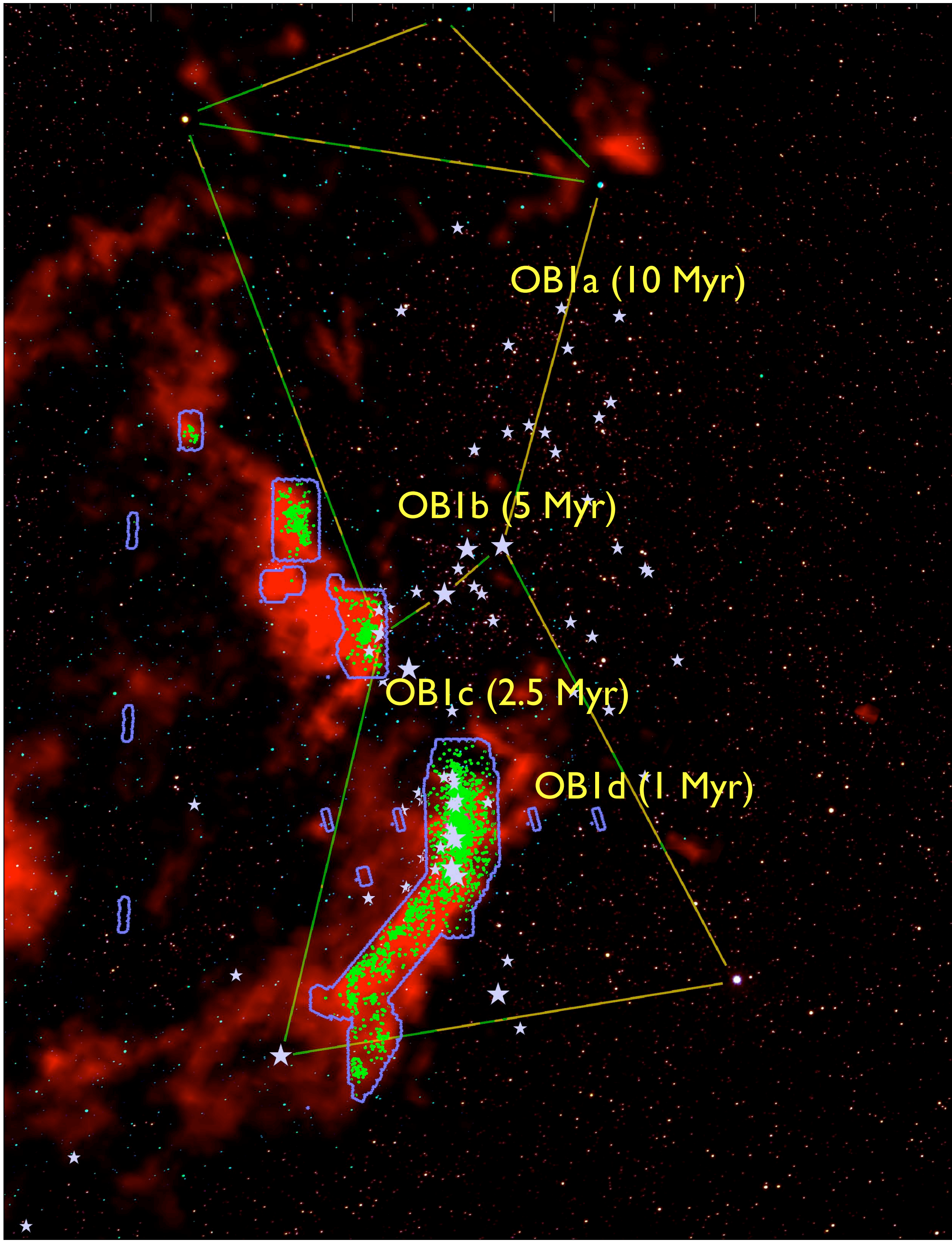
Lecture 14: Viscous Accretion Disks



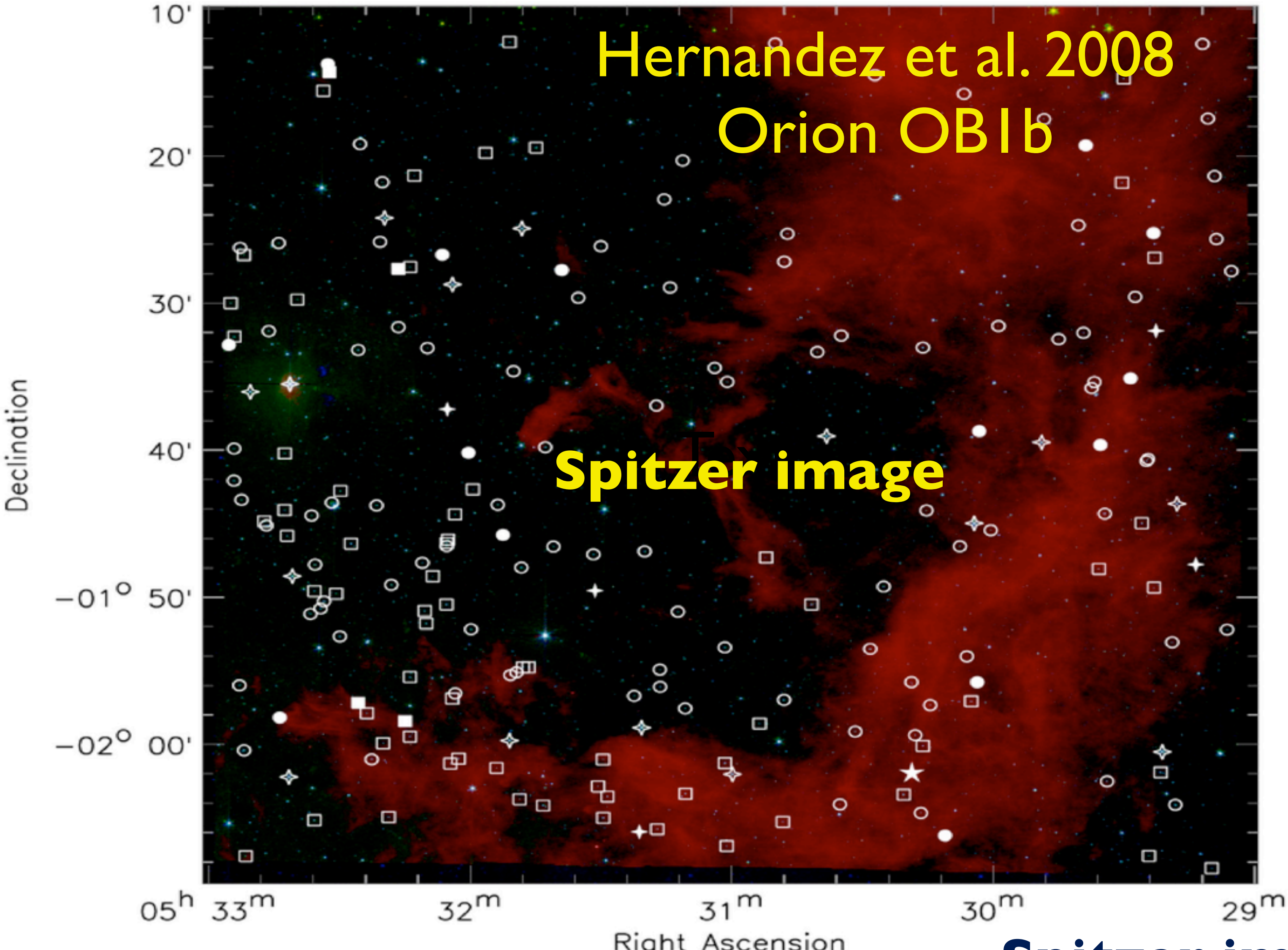
HST: 0.2 - 1 μm image of the A0e star AB Aurigae



Subgroups in the Orion OB1 Associations

The Orion OB1 association can be subdivided into four subgroups, each with a different age.

OB Associations as Laboratories for Disk Evolution



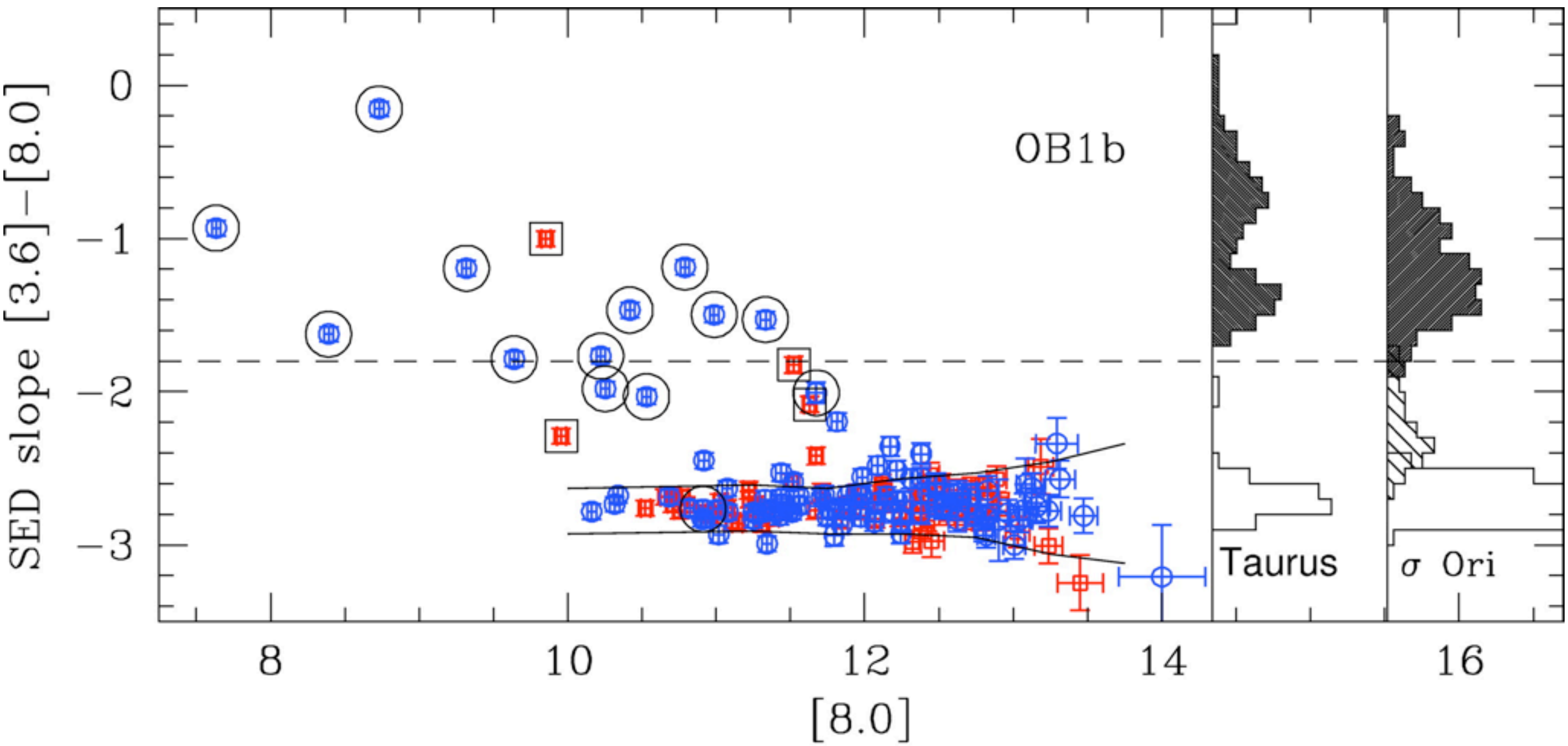
Spitzer image

Squares and Cricles: Members
Filled squares and cirles: sources with disks

OB Associations as Laboratories for Disk Evolution

Hernandez et al. 2008

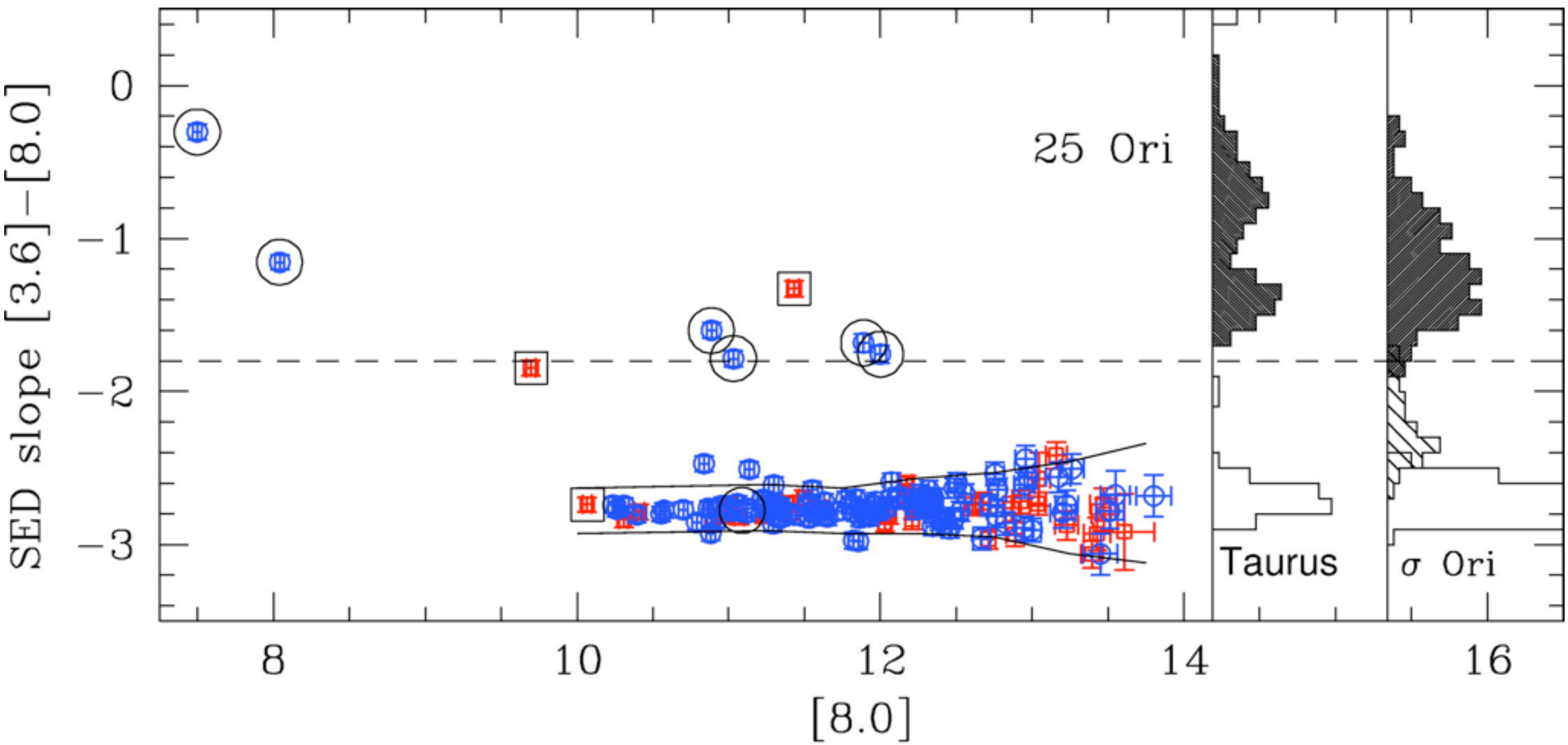
OB1b: 5 Myr



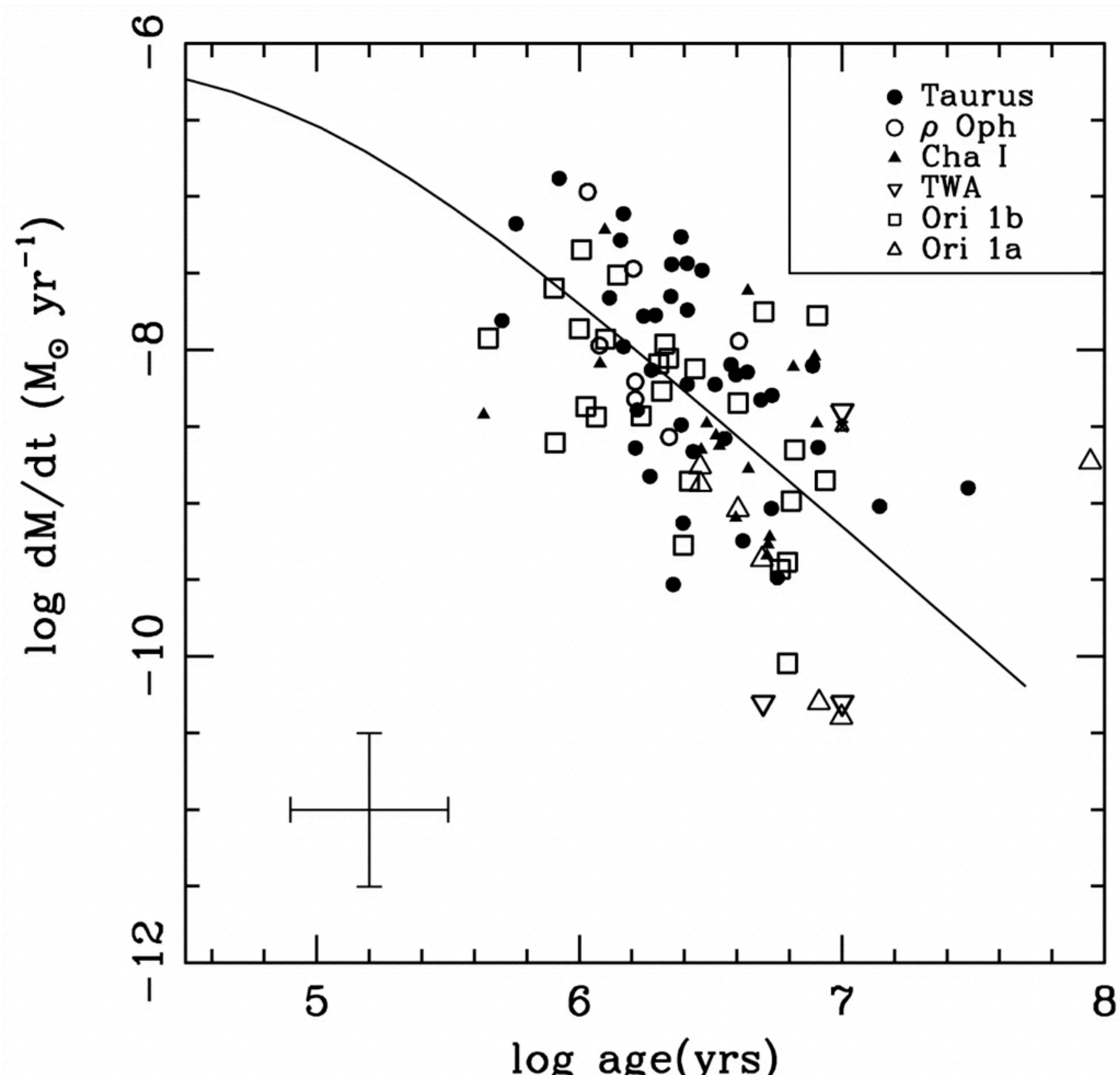
OB Associations as Laboratories for Disk Evolution

Hernandez et al. 2008

OB1a: 10 Myr

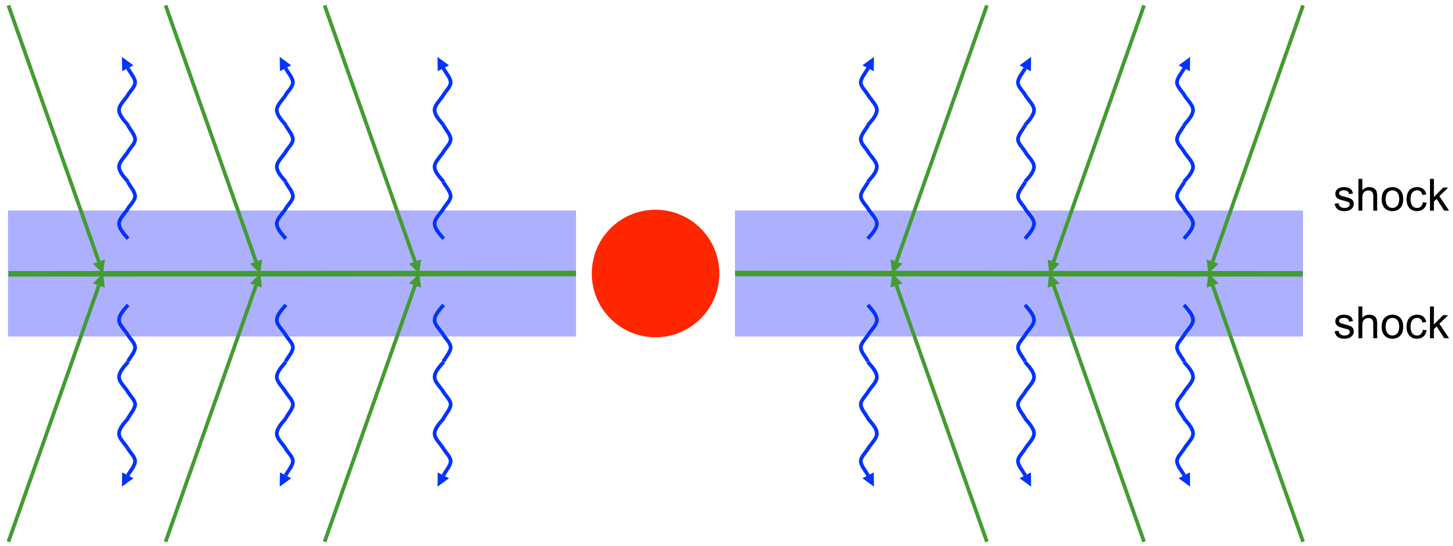


OB Associations as Laboratories for Disk Evolution



Calvet et al. 2005

The formation of a disk



- Infalling matter collides with matter from the other side
- Forms a shock
- Free-fall kinetic energy is converted into heat

$$\frac{kT}{\mu m_p} \approx \tfrac{1}{2} v_{\rm ff}^2 = \frac{GM_*}{r}$$

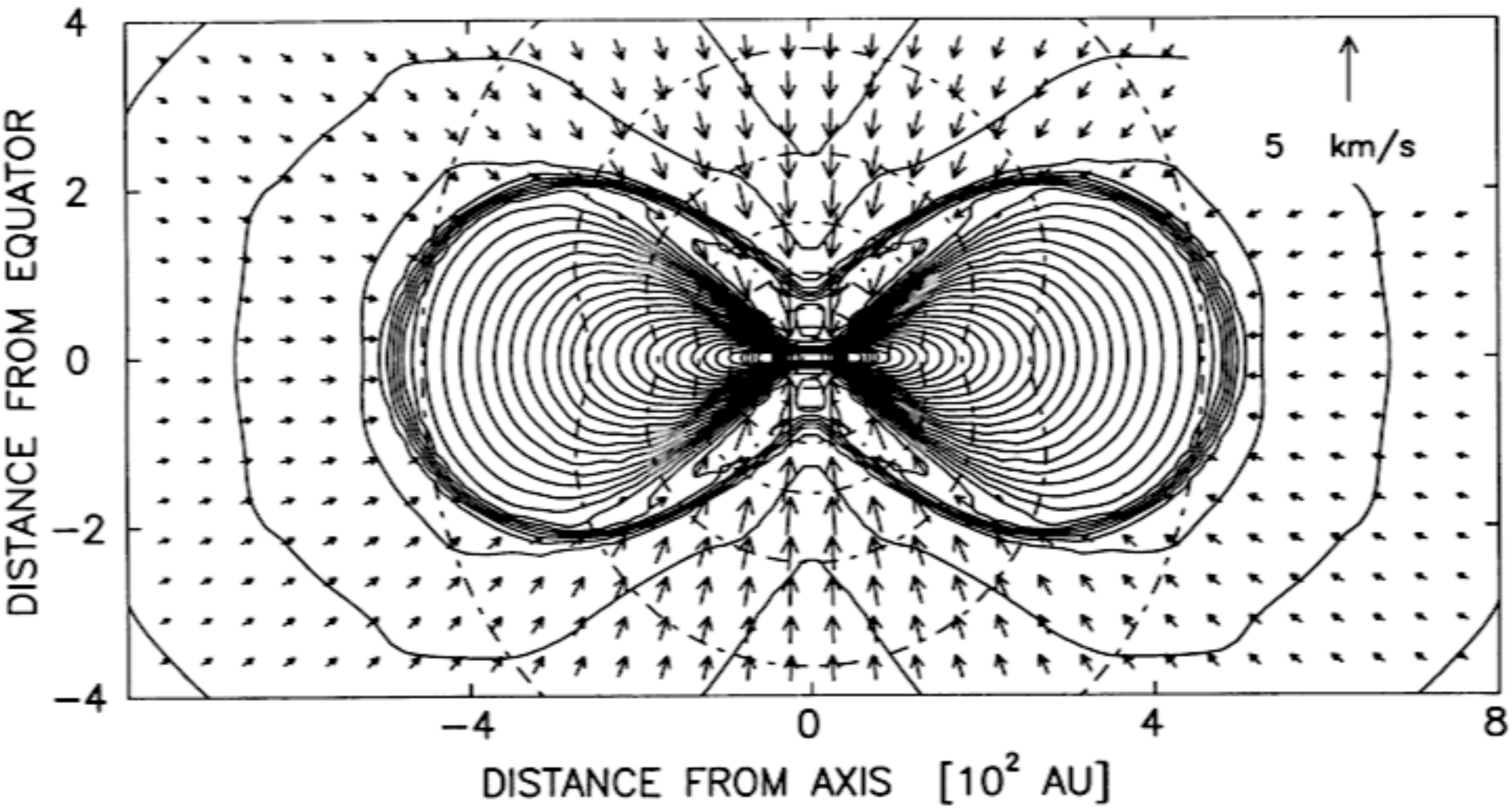
At 10 AU from $1M_\odot$ star:

$$T \approx 25000\,K$$

- Heat is radiated away, matter cools, sediments to midplane
- Disk is formed

The formation of a disk

3-D Radiation-Hydro simulations of disk formation



Yorke, Bodenheimer & Laughlin 1993

Keplerian rotation (a reminder)

Disk material is almost (!) 100% supported against gravity by its rotation. Gas pressure plays only a minor role. Therefore it is a good approximation to say that the tangential velocity of the gas in the disk is:

$$v_\phi \cong \Omega_K r = \sqrt{\frac{GM_*}{r}}$$

$$\Omega_K \equiv \sqrt{\frac{GM_*}{r^3}}$$

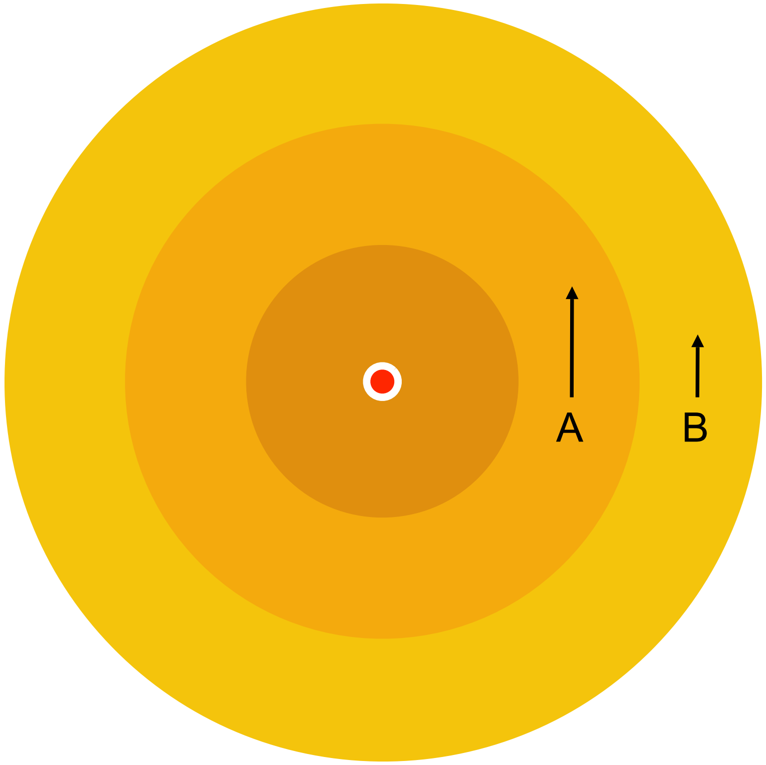
Kepler frequency

The angular momentum problem

- Angular momentum of 1 $M_\odot$ in 10 AU disk:
  $3\text{x}10^{53}$ cm^2/s
- Angular momentum of 1 $M_\odot$ in 1 $R_\odot$ star:
  <<$6\text{x}10^{51}$ cm^2/s (=breakup-rotation-speed)
- Original angular momentum of disk = 50x higher than maximum allowed for a star
- Angular momentum is strictly conserved!
- Two possible solutions:
  - Torque against external medium (via magnetic fields?)
  - Very outer disk absorbs all angular momentum by moving outward, while rest moves inward: Need *friction through viscosity*!

Outward angular momentum transport

Ring A moves faster than ring B. Friction between the two will try to slow down A and speed up B. This means: angular momentum is transferred from A to B.

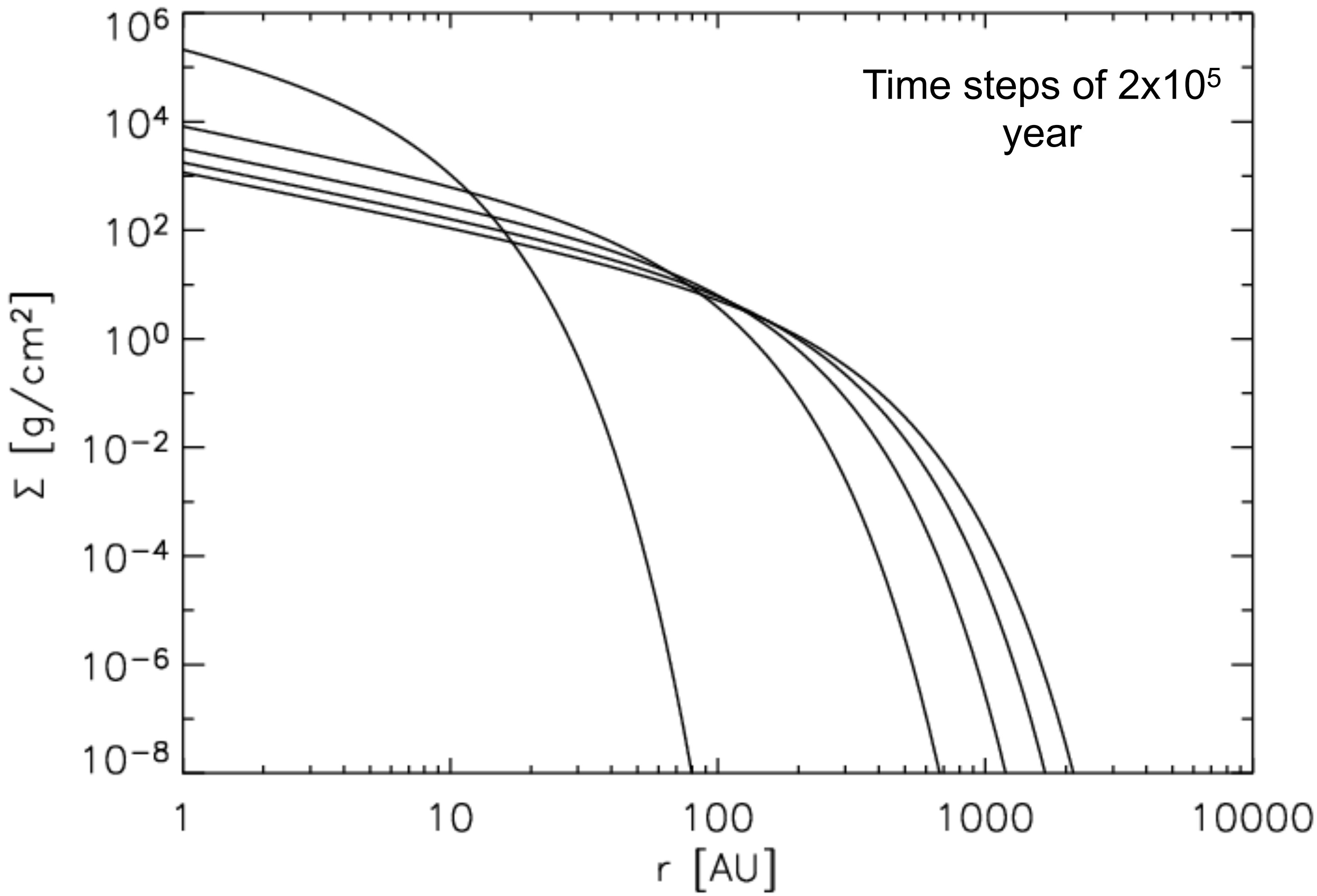


Specific angular momentum for a Keplerian disk:

$$l = r v_\phi = r^2 \Omega_K = \sqrt{G M_* r}$$

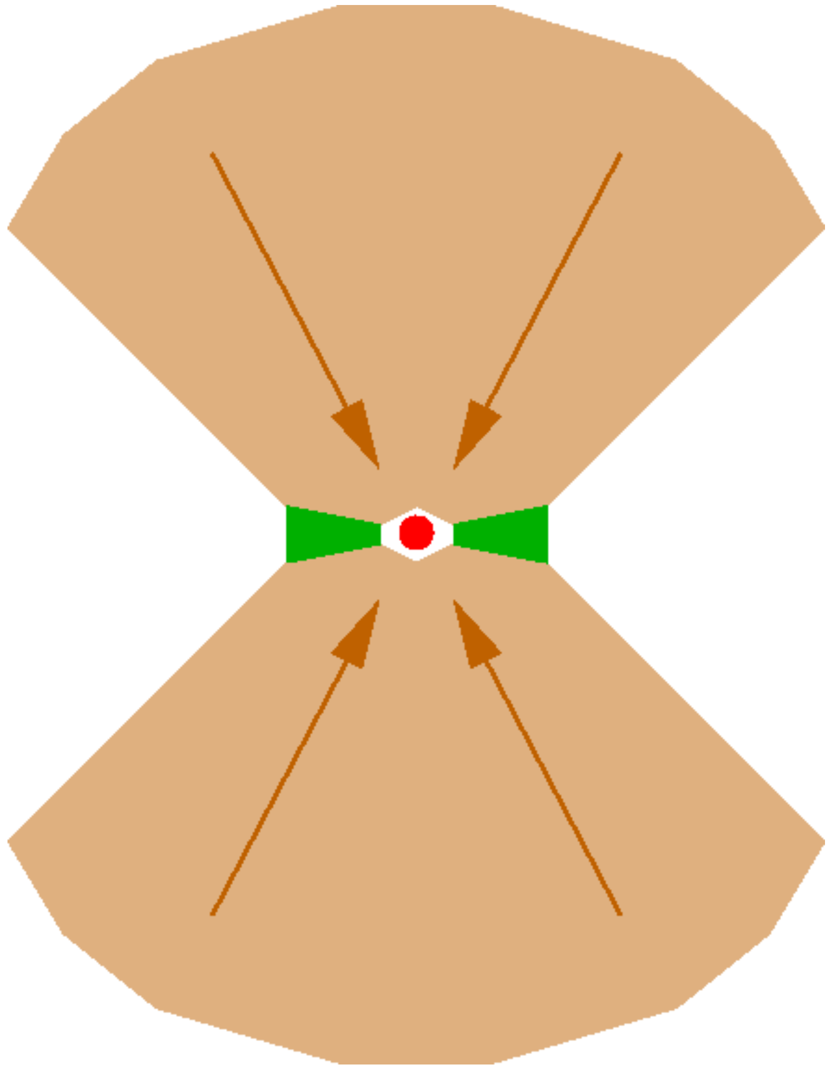
So if ring A looses angular momentum, but is forced to remain on a Kepler orbit, it must move inward! Ring B moves outward, unless it, too, has friction (with a ring C, which has friction with D, etc.).

Non-stationary (spreading) disks

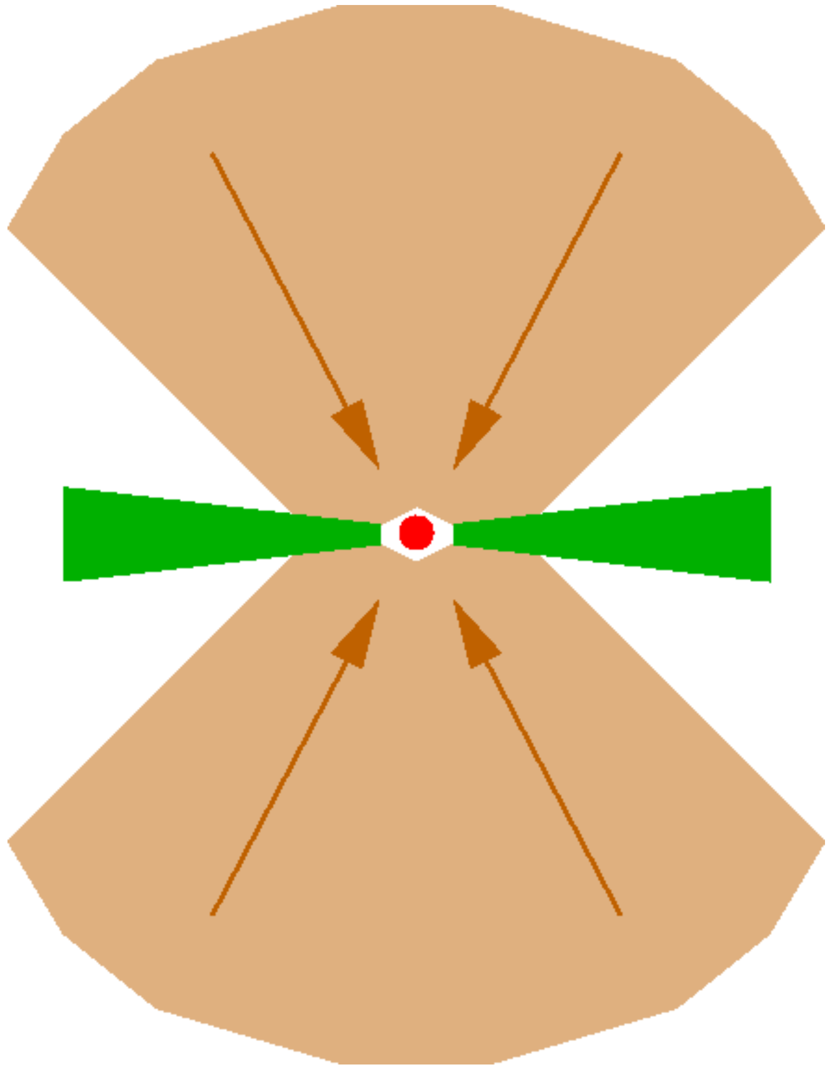


Lynden-Bell & Pringle (1974), Hartmann et al. (1998)

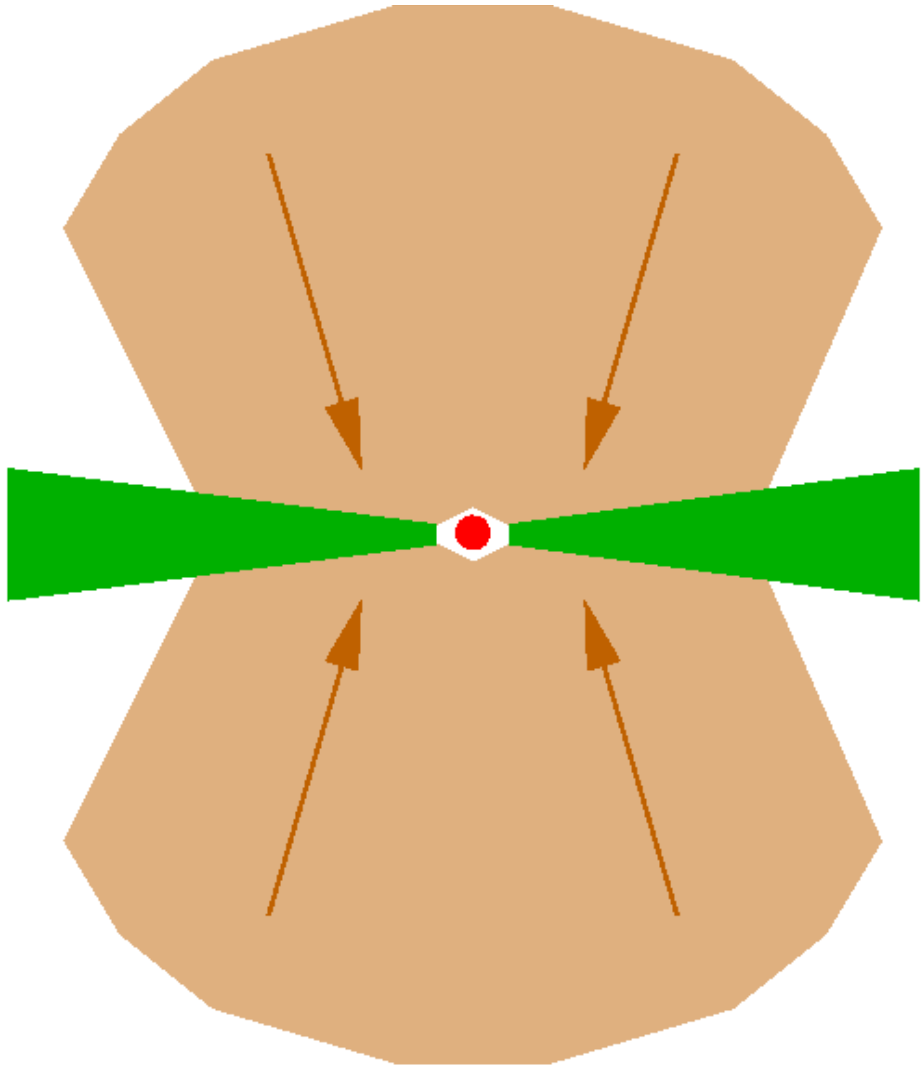
Formation & viscous spreading of disk



Formation & viscous spreading of disk



Formation & viscous spreading of disk



Formation & viscous spreading of disk

From the rotating collapsing cloud model we know:

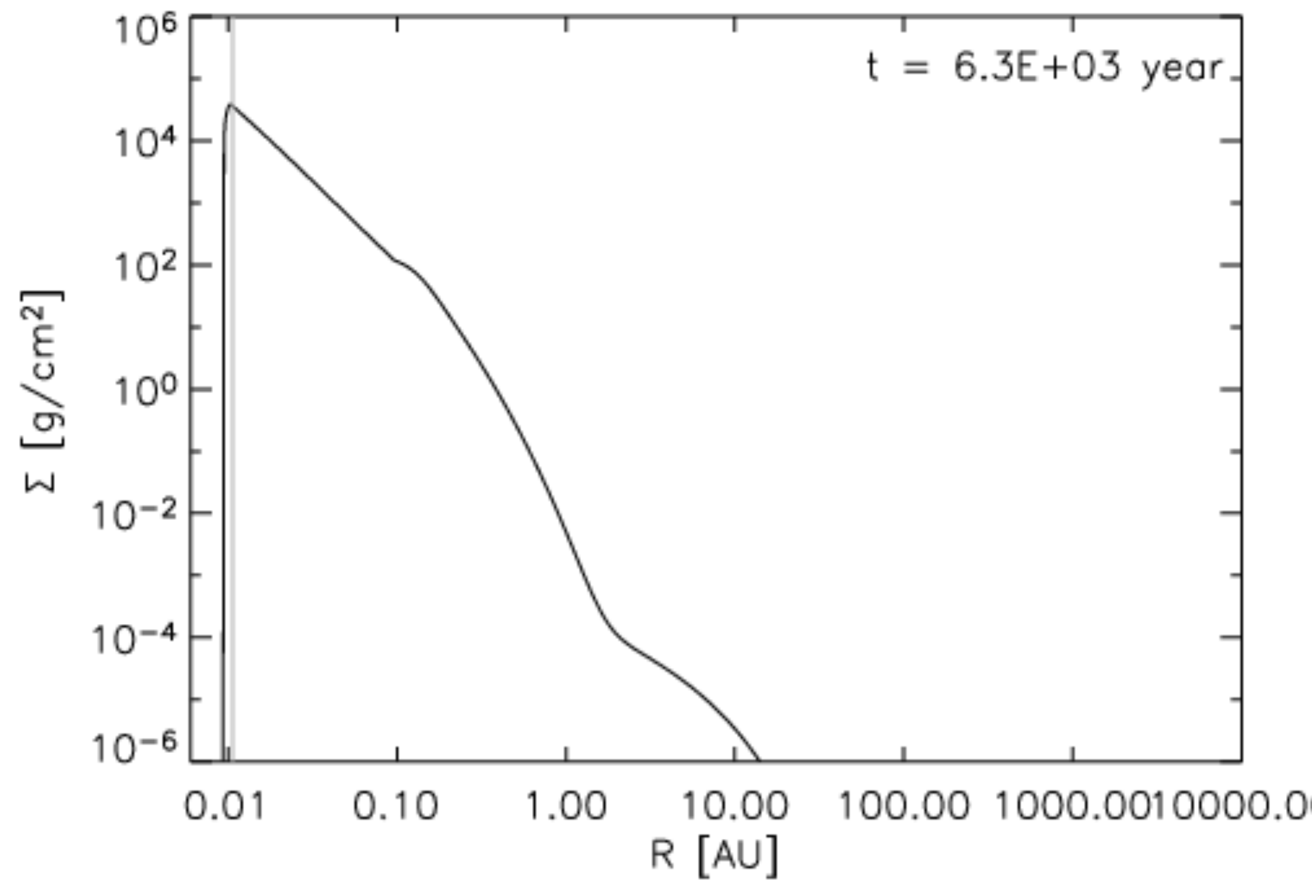
$$r_{\rm centrif} \sim t^4$$

Initially the disk spreads faster than the centrifugal radius.

Later the centrifugal radius increases faster than disk spreading

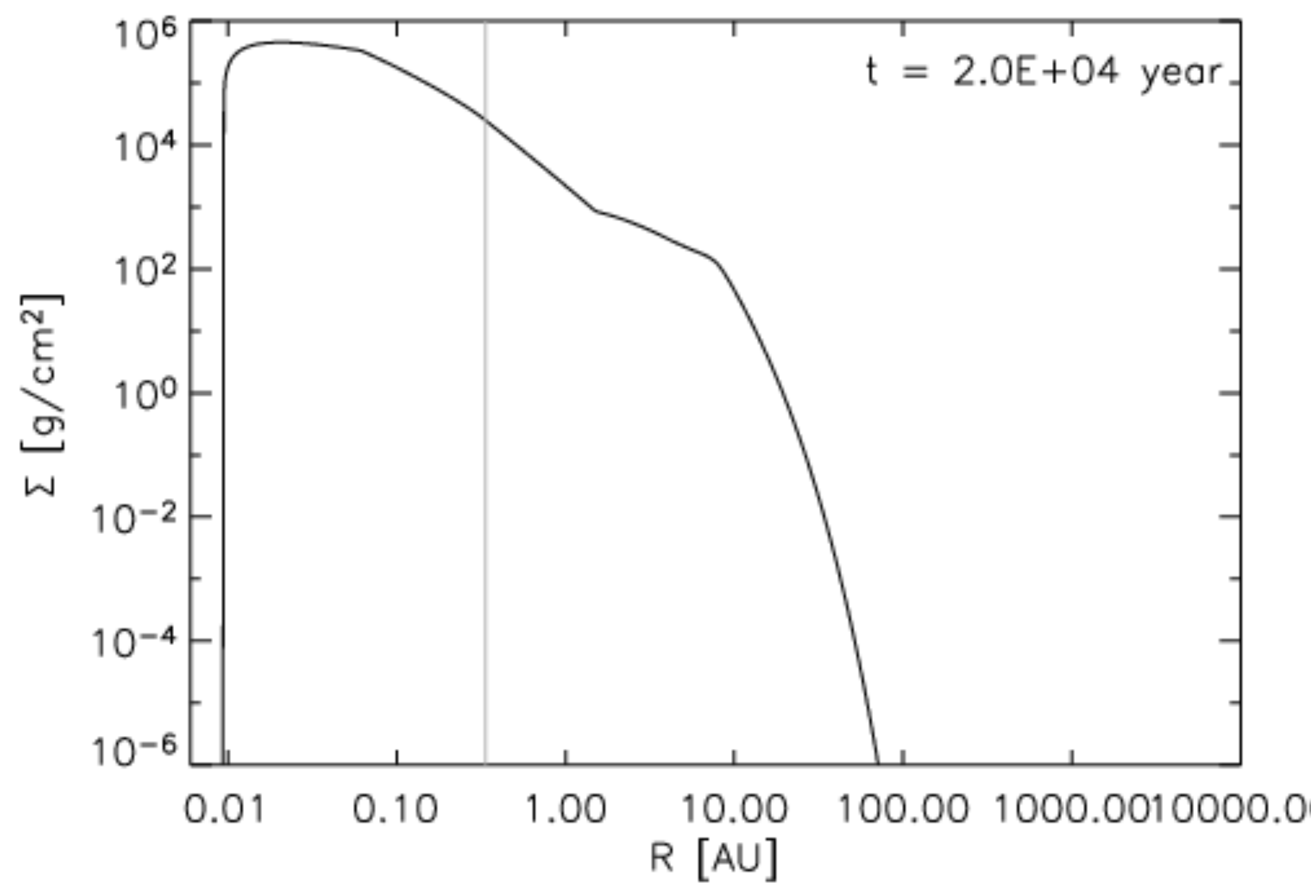
Formation & viscous spreading of disk

A numerical model



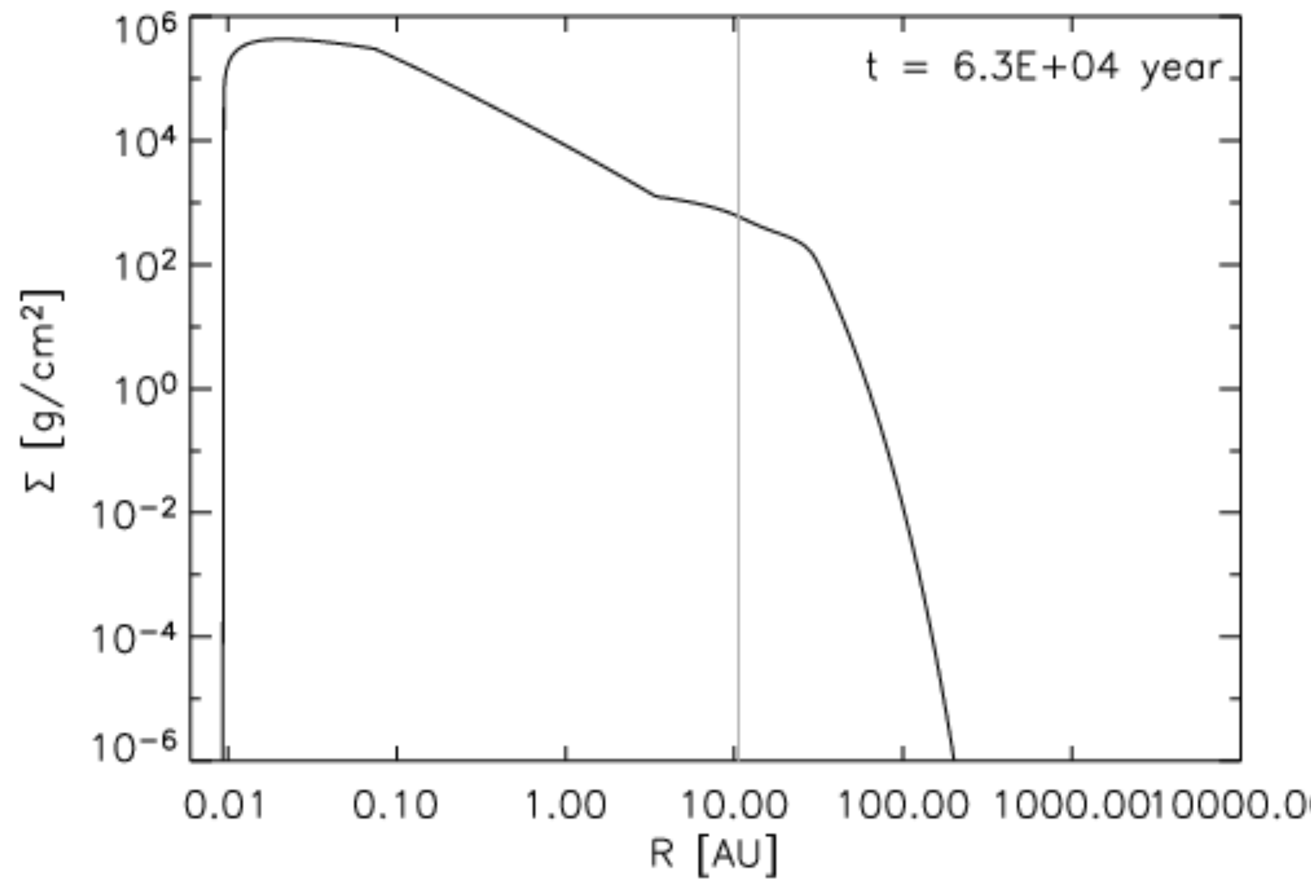
Formation & viscous spreading of disk

A numerical model



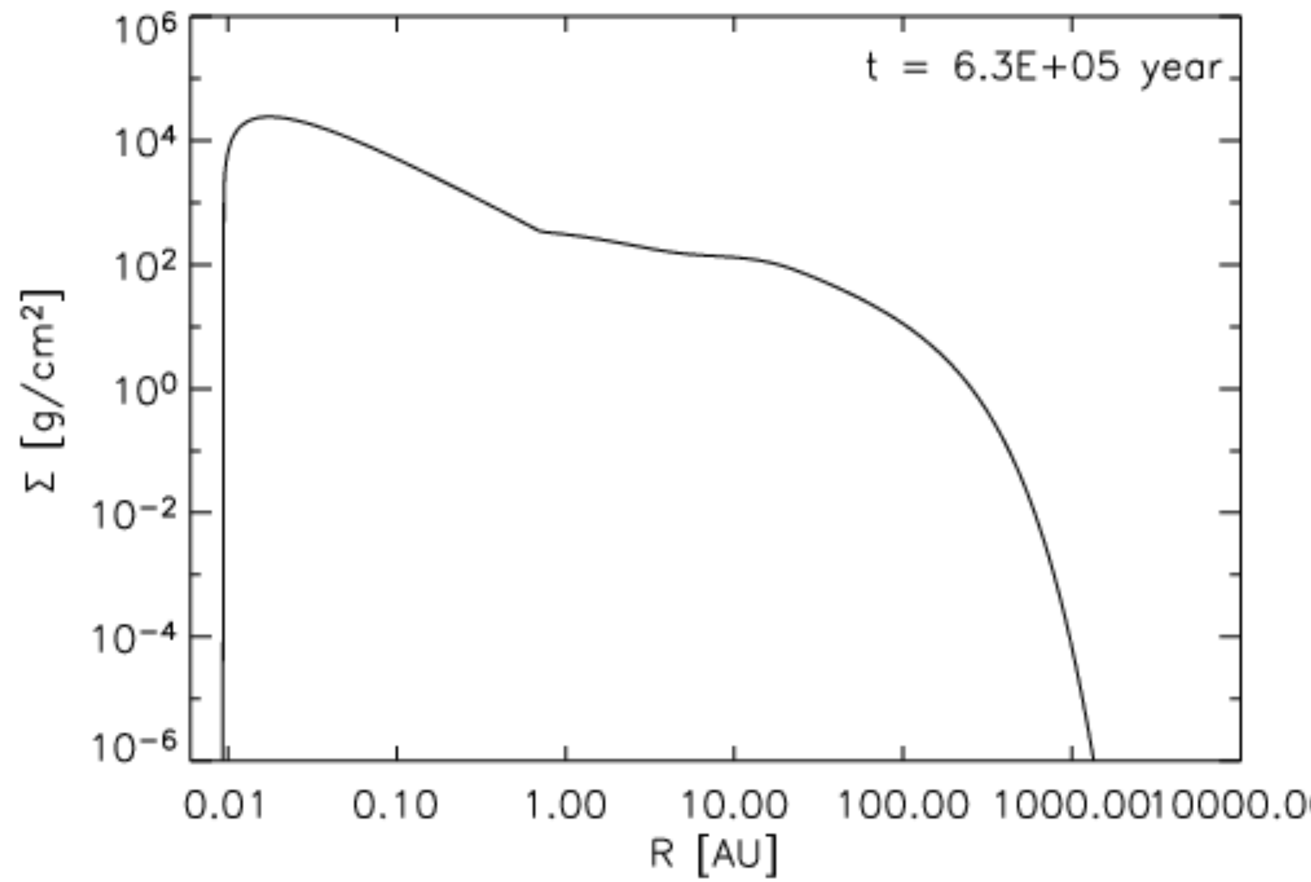
Formation & viscous spreading of disk

A numerical model



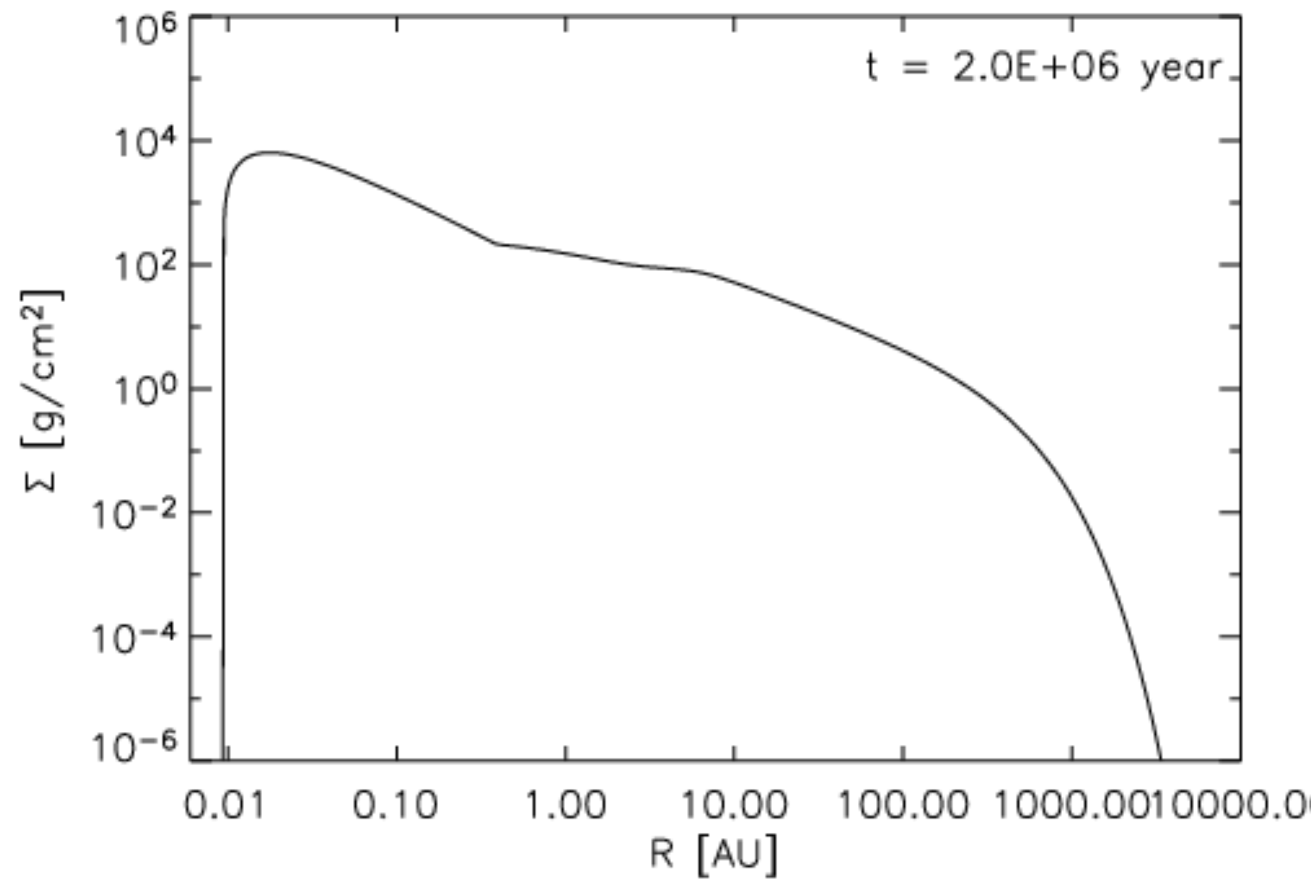
Formation & viscous spreading of disk

A numerical model

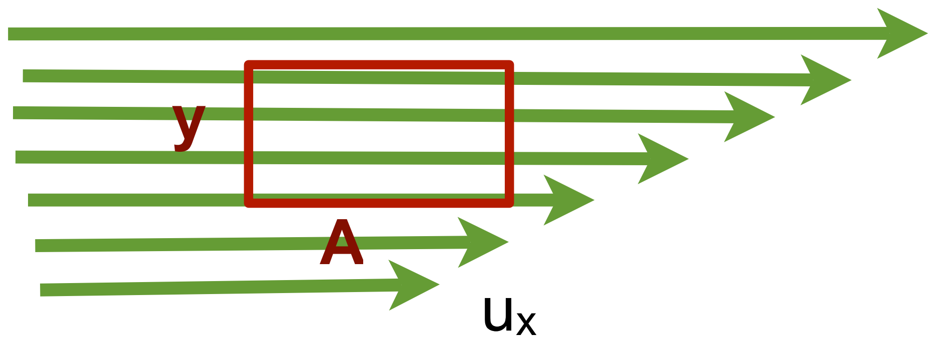


Formation & viscous spreading of disk

A numerical model



Viscosity



$$\frac{dp_x}{dt} = l_{mfp} v_{th} \rho \frac{\partial u_x}{\partial y} A \quad (1)$$

where l_{mfp} is the mean free path, which is equal to $1/(n\sigma)$ and v_{th} is the mean velocity from thermal motions.

$$l_{mfp} = \frac{1}{n\sigma} = \frac{\mu m_H}{\rho\sigma}, v_{th} = c_s = \sqrt{\frac{kT}{\mu m_H}} \quad (2)$$

giving

$$\frac{dp_x}{dt} = \frac{\mu m_H c_s}{\sigma} \frac{\partial u_x}{\partial y} A \quad (3)$$

Viscosity

Now in a constant velocity sheer, each layer receives as much momentum from the upper layer as it loses to the lower layers. Show a change in velocity, or a

$$\frac{dp_x}{dt} = \frac{\partial}{\partial y}\left(\frac{\mu m_H c_s}{\sigma}\frac{\partial u_x}{\partial y}\right) A\Delta y \tag{4}$$

we can divide out $A\Delta y$ to determine the force unit per volume

$$f_x = \frac{\partial}{\partial y}\left(\frac{\mu m_H c_s}{\sigma}\frac{\partial u_x}{\partial y}\right) \tag{5}$$

this can be written as

$$f_x = \frac{\partial \pi_{xy}}{\partial y},\ where\ \pi_{xy} = \rho\nu\frac{\partial u_x}{\partial y} \tag{6}$$

$$\nu = \frac{\mu m_H c_s}{\rho\sigma} = l_{mfp} c_s \tag{7}$$

Shakura & Sunyaev model

Difficulty: re-express equations in cylindrical coordinates. Complex due to covariant derivatives of tensors... You'll have to simply believe the Equations I write here...

Radial structure

Define the surface density:

$$\Sigma(r) \equiv \int_{-\infty}^{+\infty} \rho(r,z)\, dz$$

Integrate continuity equation over z:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial(\Sigma r v_r)}{\partial r} = 0 \qquad (1)$$

Integrate radial momentum equation over z:

$$\frac{\partial(\Sigma \mathrm{v}_\mathrm{r})}{\partial t} + \frac{1}{r}\frac{\partial(\Sigma r \mathrm{v}_r^2)}{\partial r} + \frac{\partial(\Sigma \mathrm{c}_\mathrm{s}^2)}{\partial r} - \Sigma\frac{\mathrm{v}_\phi^2}{r} = -\Sigma\frac{GM}{r^2} \qquad (2)$$

Integrate tangential momentum equation over z:

$(l \equiv \mathrm{v}_\phi r)$
$(\Omega \equiv \mathrm{v}_\phi / r)$

$$\frac{\partial(\Sigma l)}{\partial t} + \frac{1}{r}\frac{\partial(\Sigma r\, \mathrm{v}_\mathrm{r} l)}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(\Sigma \nu\, r^3 \frac{\partial \Omega}{\partial r}\right) \qquad (3)$$

Shakura & Sunyaev model

Let's first look closer at the radial momentum equation:

$$\cancel{\frac{\partial(\Sigma \mathrm{v}_r)}{\partial t}} + \cancel{\frac{1}{r}\frac{\partial(\Sigma r \mathrm{v}_r^2)}{\partial r}} + \cancel{\frac{\partial(\Sigma c_s^2)}{\partial r}} - \Sigma\frac{\mathrm{v}_\phi^2}{r} = -\Sigma\frac{GM}{r^2} \qquad (2)$$

Let us take the Ansatz (which one can later verify to be true) that $\mathrm{v}_r << c_s << \mathrm{v}_\phi$.

$$\mathrm{v}_\phi^2 = \frac{GM}{r}$$

$$\boxed{\mathrm{v}_\phi = \sqrt{\frac{GM}{r}} \equiv \Omega_K r}$$

That means: from the *radial* momentum equation follows the *tangential* velocity

$\boxed{\text{Conclusion: the disk is Keplerian}}$

Shakura & Sunyaev model

Let's now look closer at the tangential momentum equation:

$$\frac{\partial(\Sigma l_K)}{\partial t} + \frac{1}{r}\frac{\partial(\Sigma r \mathrm{v}_r l_K)}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(\Sigma \nu r^3 \frac{\partial \Omega_K}{\partial r}\right) \qquad (3)$$

Now use continuity equation

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial(\Sigma r \mathrm{v}_r)}{\partial r} = 0$$

$$\Sigma \mathrm{v}_r \frac{\partial(\Omega_K r^2)}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(\Sigma \nu r^3 \frac{\partial \Omega_K}{\partial r}\right) \qquad l_K \equiv \Omega_K r^2$$

The derivatives of the Kepler frequency can be worked out:

$$\longrightarrow \quad \boxed{\mathrm{v}_r = -\frac{3}{\Sigma\sqrt{r}}\frac{\partial}{\partial r}\left(\Sigma \nu \sqrt{r}\right)}$$

That means: from the *tangential* momentum equation follows the *radial* velocity

Shakura & Sunyaev model

Radial structure

Our radial structure equations have now reduced to:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial(\Sigma r \mathrm{v}_r)}{\partial r} = 0 \qquad \text{with} \qquad \mathrm{v}_\mathrm{r} = -\frac{3}{\Sigma\sqrt{r}}\frac{\partial}{\partial r}\left(\Sigma \nu \sqrt{r}\right)$$

Missing piece: what is the value of ν?

It is not really known what value ν has. This depends on the details of the source of viscosity. But from dimensional analysis it must be something like:

$$\nu = \alpha c_s h \qquad \alpha = 0.001 \ldots 0.1$$

Alpha-viscosity (Shakura & Sunyaev 1973)

Shakura & Sunyaev model

Further on alpha-viscosity:

$$\nu = \alpha c_s h$$

Here the vertical structure comes back into the radial structure equations!

$$h = \sqrt{\frac{kTr^3}{\mu m_p G M_*}} \equiv \frac{c_s}{\Omega_K}$$

So we obtain for the viscosity:

$$\boxed{\nu = \alpha \frac{c_s^2}{\Omega_K}}$$

Shakura & Sunyaev model

Summary of radial structure equations:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial(\Sigma r \mathrm{v}_r)}{\partial r} = 0$$

$$\mathrm{v}_{\mathrm{r}} = -\frac{3}{\Sigma\sqrt{r}}\frac{\partial}{\partial r}\left(\Sigma \nu \sqrt{r}\right)$$

$$\nu = \alpha \frac{c_s^2}{\Omega_K}$$

<u>If we know</u> the temperature c_s^2 everywhere, we can readily solve these equations (time-dependent or stationary, whatever we like).

<u>If we don't know</u> the temperature a-priori, then we need to solve the above 3 equations simultaneously with energy equation.

Shakura & Sunyaev model

<u>Suppose we *know* that c_s^2 is a given power-law:</u>

Ansatz: surface density is also a powerlaw:

$$c_s^2 \sim r^{-\xi} \qquad\qquad \nu \equiv \alpha \frac{c_s^2}{\Omega_K} \sim r^{-\xi+3/2}$$

$$\Sigma \sim r^{-\eta}$$

The radial velocity then becomes:

$$\mathrm{v}_\mathrm{r} \equiv -\frac{3}{\Sigma\sqrt{r}}\frac{\partial}{\partial r}\left(\Sigma\nu\sqrt{r}\right) = 3(\xi+\eta-2)\frac{\nu}{r} \longrightarrow \boxed{\mathrm{v}_\mathrm{r} = -\frac{3}{2}\frac{\nu}{r}}$$

Stationary continuity equation:

$$\frac{\partial(\Sigma r \mathrm{v}_r)}{\partial r} = 0$$

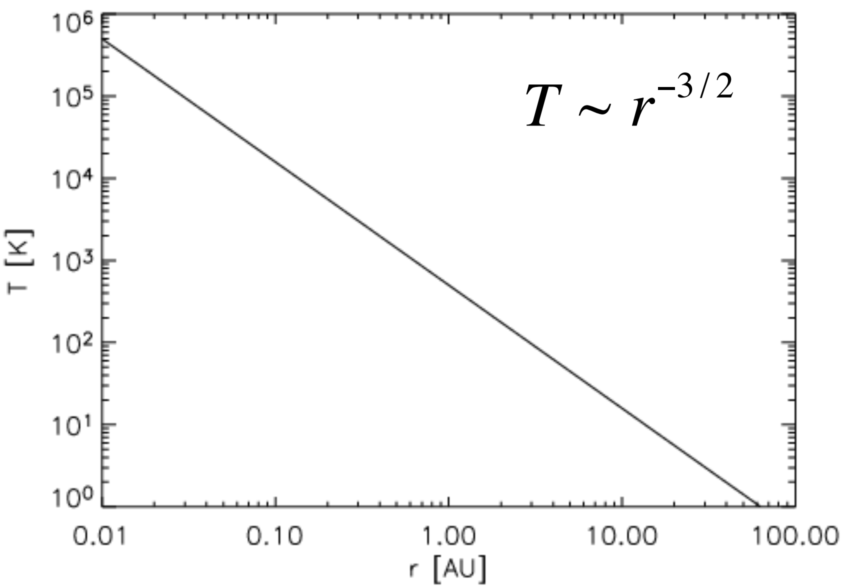
from which follows:

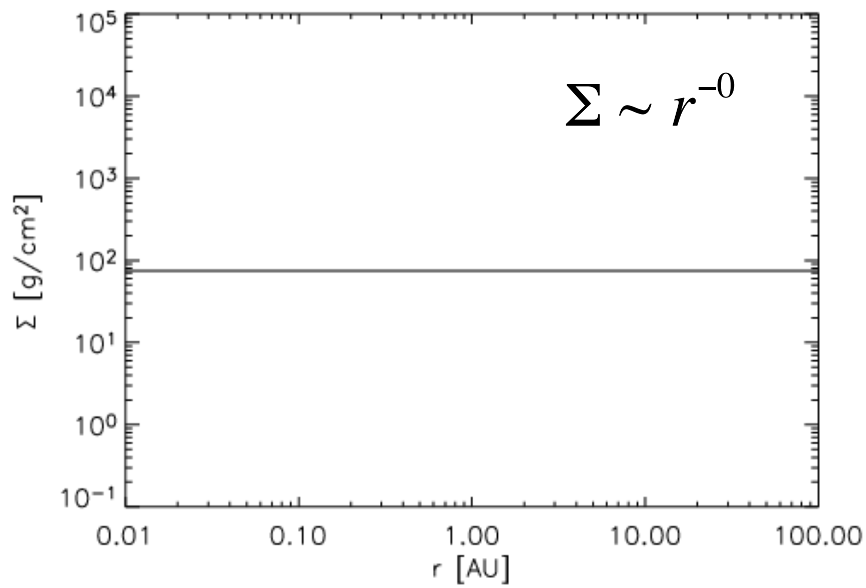
$$\boxed{\Sigma \sim r^{\xi-3/2}} \qquad\qquad \eta = -\xi + 3/2$$

Proportionality constants are straightforward from here on...

Shakura & Sunyaev model

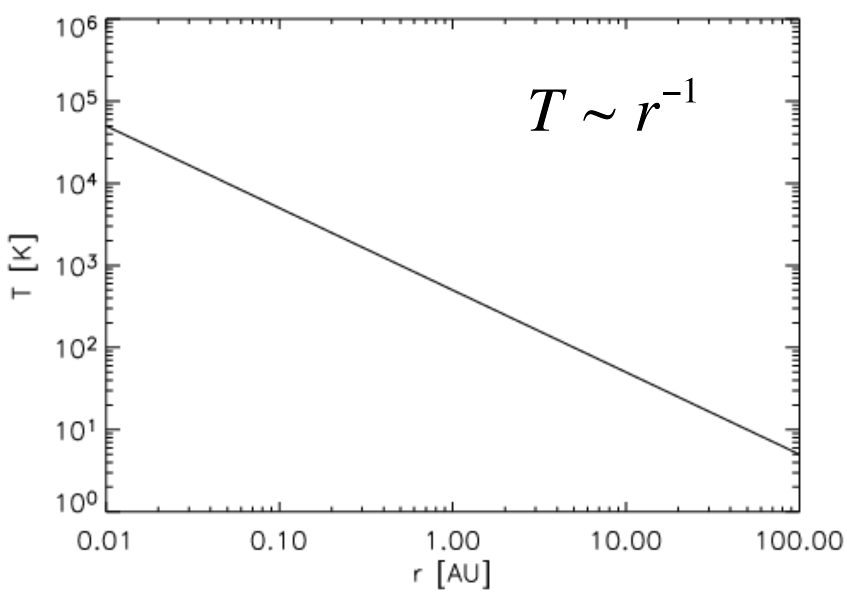
Examples:

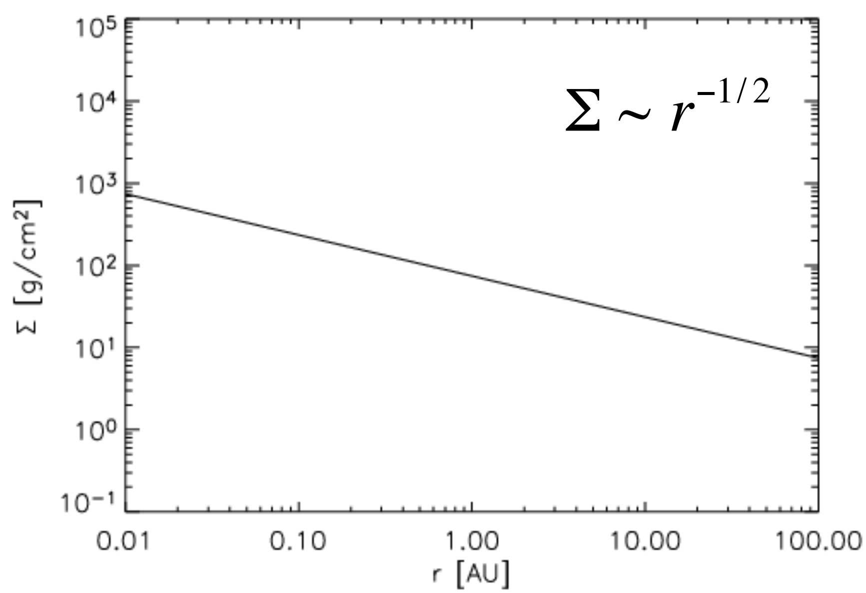




Shakura & Sunyaev model

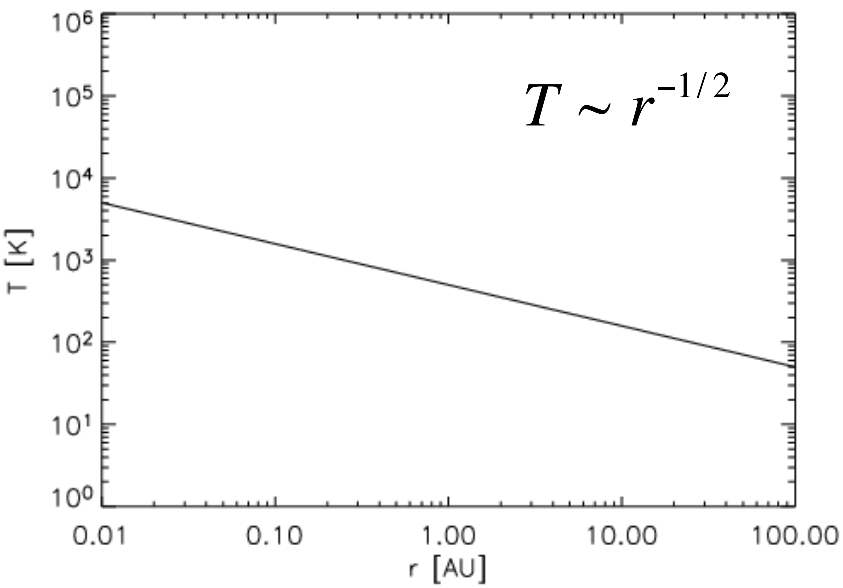
Examples:

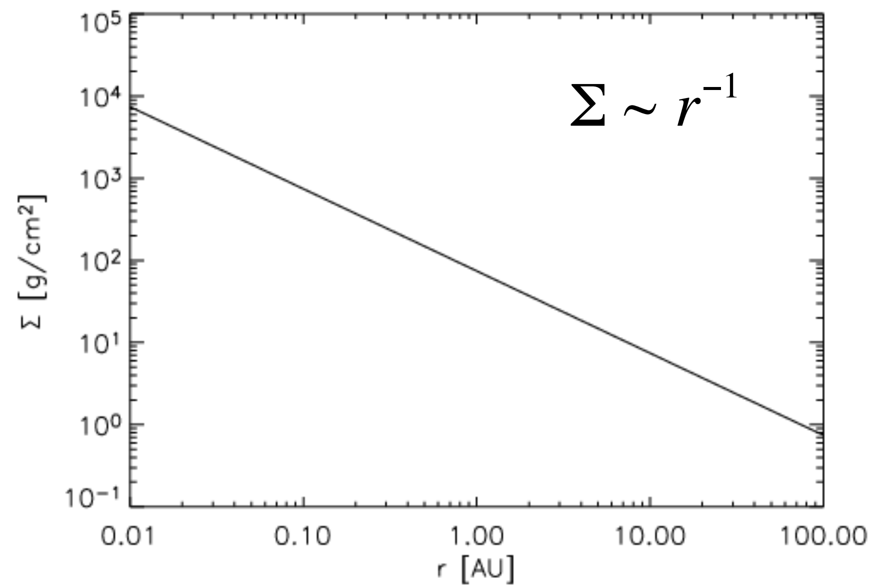




Shakura & Sunyaev model

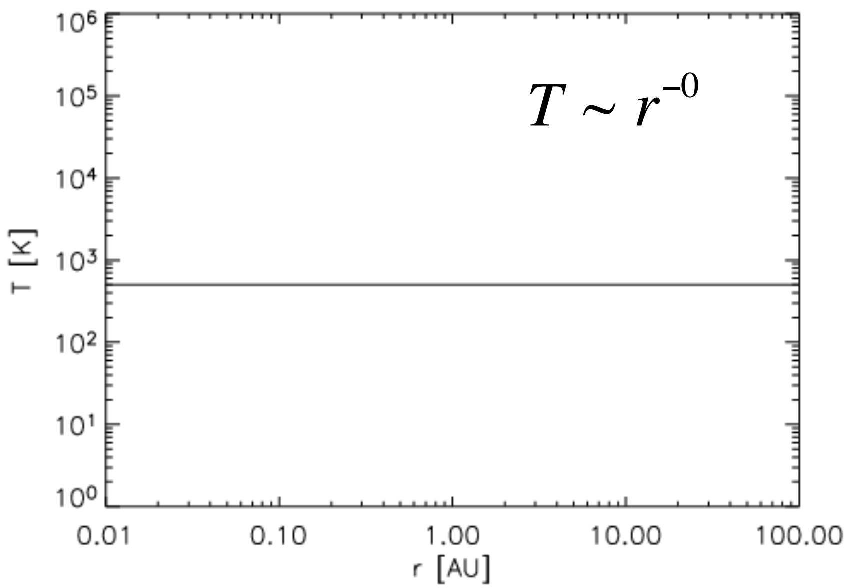
Examples:

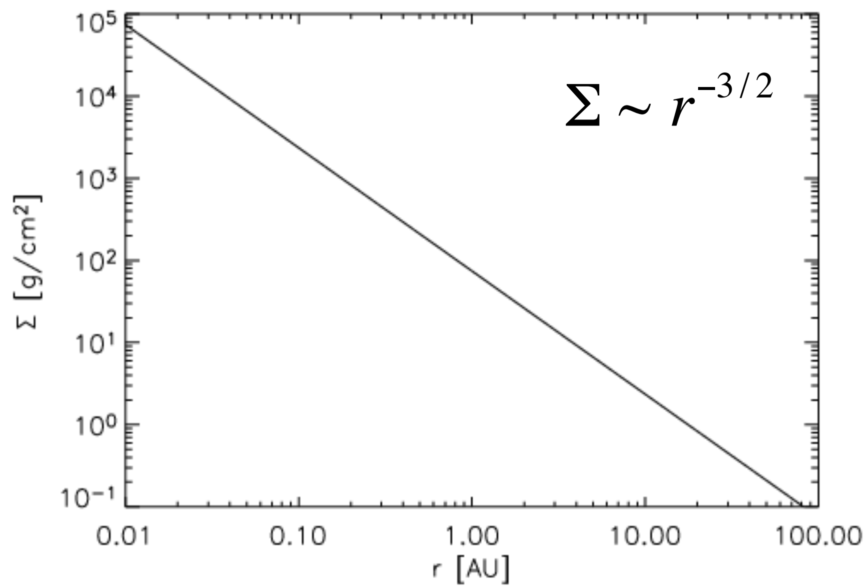




Shakura & Sunyaev model

Examples:





Shakura & Sunyaev model

Formulation in terms of accretion rate

Accretion rate $\dot{M}$ is amount of matter per second that moves radially inward throught he disk.

$$\dot{M} = -2\pi\, r \mathrm{v}_r \Sigma$$

Working this out with previous formulae:

$$\mathrm{v}_\mathrm{r} = -\frac{3}{2}\frac{\nu}{r}$$

$$\dot{M} = 3\pi\nu\Sigma$$

$$\nu \equiv \alpha \frac{c_s^2}{\Omega_K}$$

$$\dot{M} = 3\pi\alpha \frac{c_s^2}{\Omega_K}\Sigma$$

We finally obtain:

$$\Sigma = \frac{\dot{M}}{3\pi\alpha}\frac{\Omega_K \mu m_p}{k T_\mathrm{mid}}$$

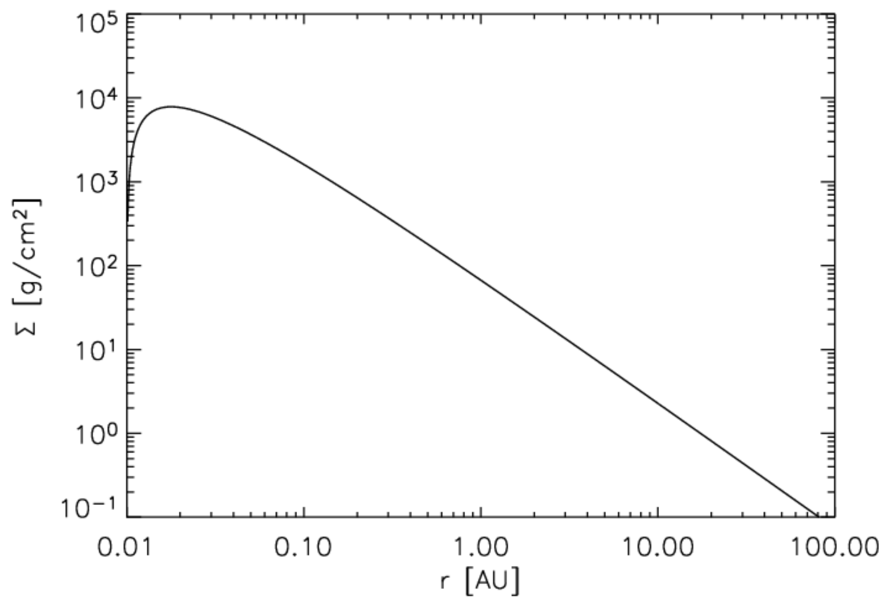
(but see later for more correct formula with inner BC satisfied)

Shakura & Sunyaev model

Effect of inner boundary condition:

Powerlaw does not go all the way to the star. At inner edge (for instance the stellar surface) there is an abrupt deviation from Keplerian rotation. This affects the structure of the disk out to many stellar radii:

$$\Sigma = \frac{\dot{M}}{3\pi\alpha}\frac{\Omega_K \mu m_p}{k T_{\rm mid}}\left[1-\left(\frac{r_{\rm in}}{\rm r}\right)^{1/2}\right]$$



Keep this in mind when applying the theory!

Shakura & Sunyaev model

How do we determine the temperature?

We must go back to the vertical structure......and the energy equation.....

First the energy equation: heat production through friction:

$$Q_+ \equiv \Sigma\nu\left(r\frac{\partial\Omega_K}{\partial r}\right)^2 = \frac{9}{4}\Sigma\nu\frac{GM_*}{r^3}$$

For power-law solution: use equation of previous page:

$$\mathrm{v_r} = -\frac{3}{2}\frac{\nu}{r} \longrightarrow \nu = -\frac{2}{3}\mathrm{r}\,\mathrm{v_r}$$

The viscous heat production becomes:

$$Q_+ = -\frac{3}{4\pi}2\pi\Sigma r\mathrm{v_r}\frac{GM_*}{r^3}$$

Shakura & Sunyaev model

How do we determine the temperature?

$$Q_+ = -\frac{3}{4\pi} 2\pi\Sigma r \mathrm{v}_\mathrm{r} \frac{GM_*}{r^3}$$

Define 'accretion rate' (amount of matter flowing through the disk per second):

$$\dot{M} \equiv -2\pi\Sigma r \,\mathrm{v}_r = \text{constant}$$

End-result for the viscous heat production:

$$\boxed{Q_+ = \frac{3}{4\pi} \dot{M}\,\Omega_K^2}$$

Shakura & Sunyaev model

How do we determine the temperature?

Now, this heat must be radiated away.

Disk has two sides, each assumed to radiate as black body:

$$2\sigma T_{\rm eff}^4 = Q_+ = \frac{3}{4\pi}\dot{M}\,\Omega_K^2$$

One obtains:

$$T_{\rm eff} = \left(\frac{3}{8\pi\sigma}\dot{M}\,\Omega_K^2\right)^{1/4} \qquad \sim r^{-3/4}$$

Molecular viscosity? No!

<u>Problem: molecular viscosity is virtually zero</u>

Reynolds number

$$\mathrm{Re} = \frac{\langle u \rangle L}{\nu}$$

L = length scale
<u>= typical velocity
ν = viscosity

Molecular viscosity:

$$\nu = \langle u_T \rangle l_{\mathrm{free}}$$

l_{free} = m.f.p. of molecule
$\langle u_T \rangle$= velo of molecule

Typical disk (at 1 AU): N=1x10^{14} cm^{-3}, T=500 K, L=0.01AU

Assume (extremely simplified) $\sigma_{H2} \approx \pi(1\mathrm{Ang})^2$.

$$\langle u_T \rangle = \sqrt{\frac{3kT}{\mu m_p}} = 2.3\,\mathrm{km/s} \qquad l_{\mathrm{free}} = \frac{1}{N\sigma} = 32\,\mathrm{cm}$$

$$\nu = 7.3 \times 10^6\ \mathrm{cm^2/s} \qquad \boxed{\mathrm{Re} = 4.7 \times 10^9}$$

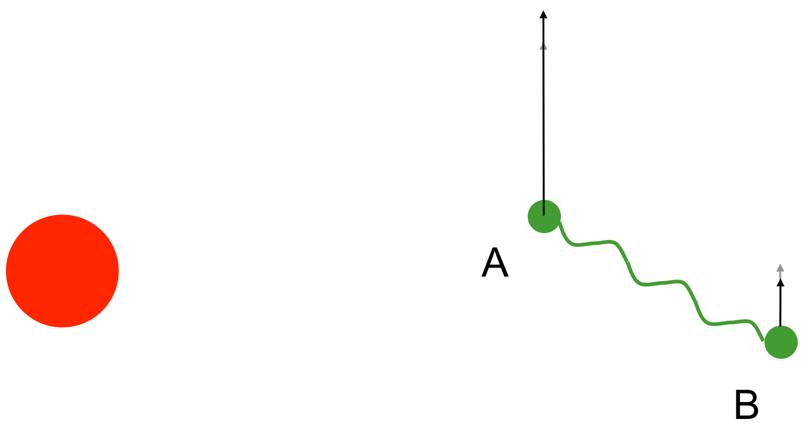
Magneto-rotational instability (MRI)

(Also often called Balbus-Hawley instability)

Highly simplified pictographic explanation:

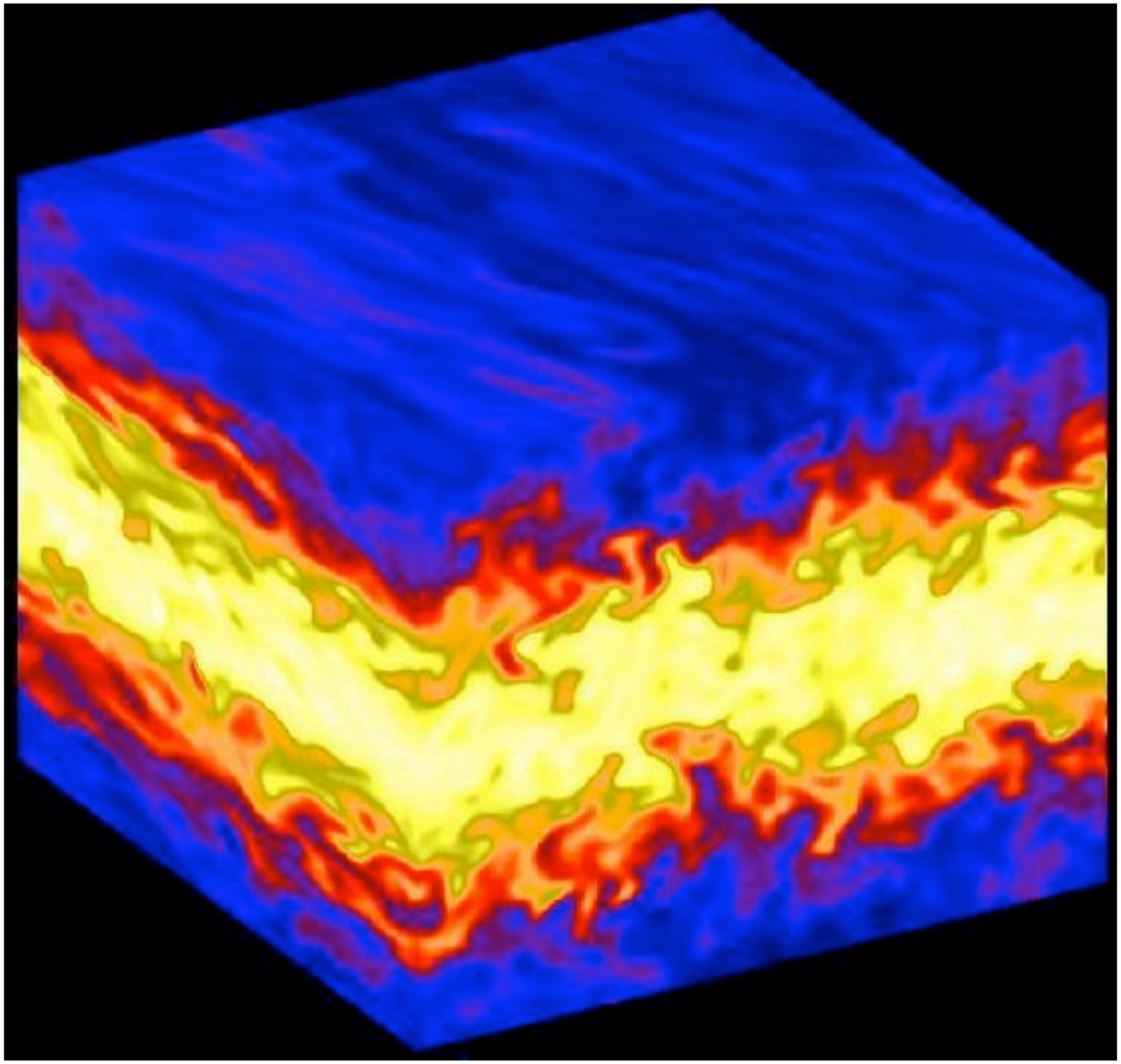
If a (weak) pull exists between two gas-parcels A and B on adjacent orbits, the effect is that A moves inward and B moves outward: a pull causes them to move apart!

The lower orbit of A causes an increase in its velocity, while B decelerates. This enhances their velocity difference! This is positive feedback: an instability.



Causes turbulence in the disk

Magneto-rotational instability (MRI)



Johansen & Klahr (2005); Brandenburg et al.

Gravitational (in)stability

If disk surface density exceeds a certain limit, then disk becomes gravitationally unstable.

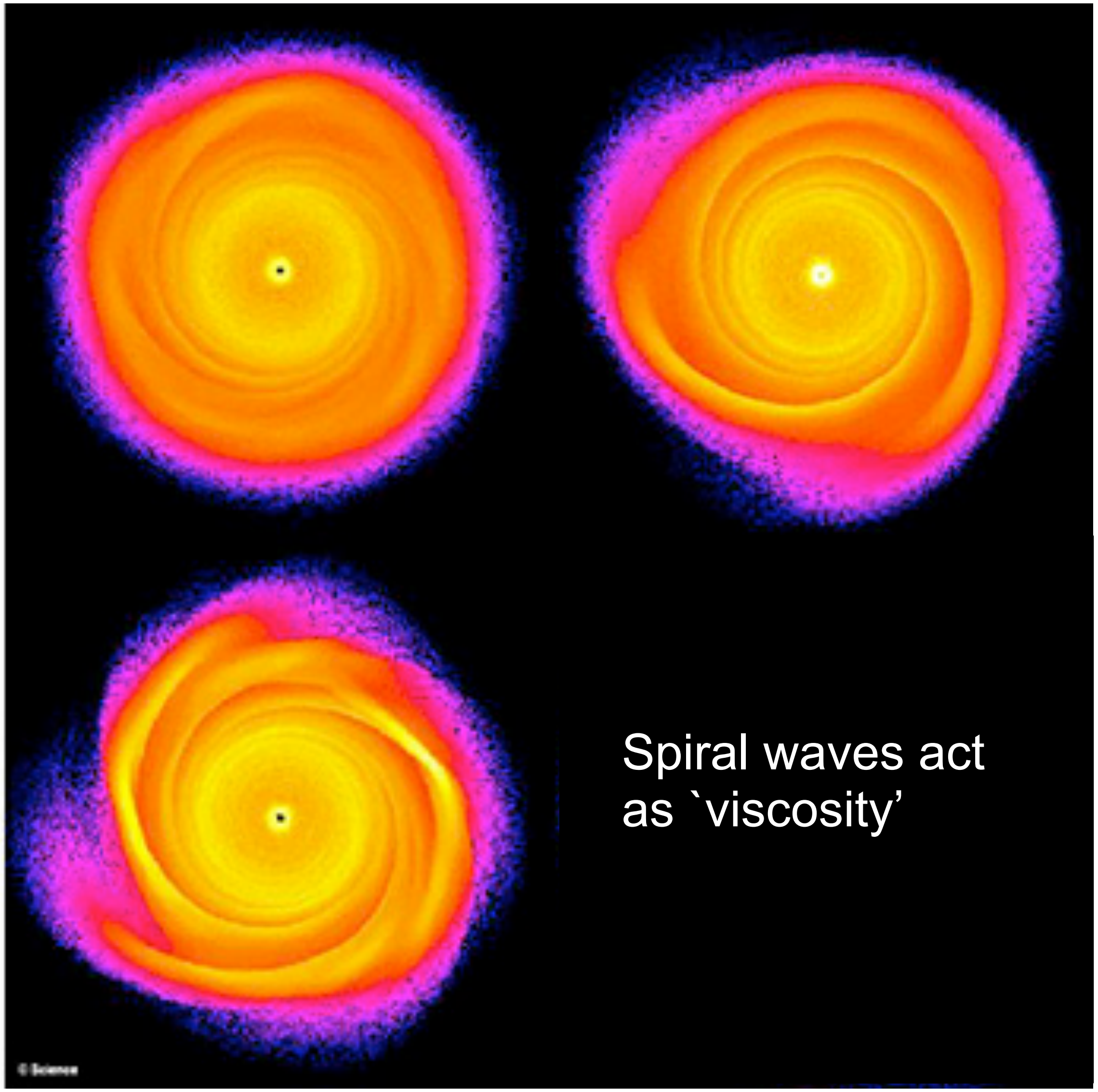
Toomre Q-parameter:

$$Q = \frac{h\,\Omega_K^2}{\pi G \Sigma} \qquad \approx \frac{h}{r}\,\frac{M_*}{M_{\rm disk}}$$

For Q>2 the disk is stable
For Q<2 the disk is gravitationally unstable

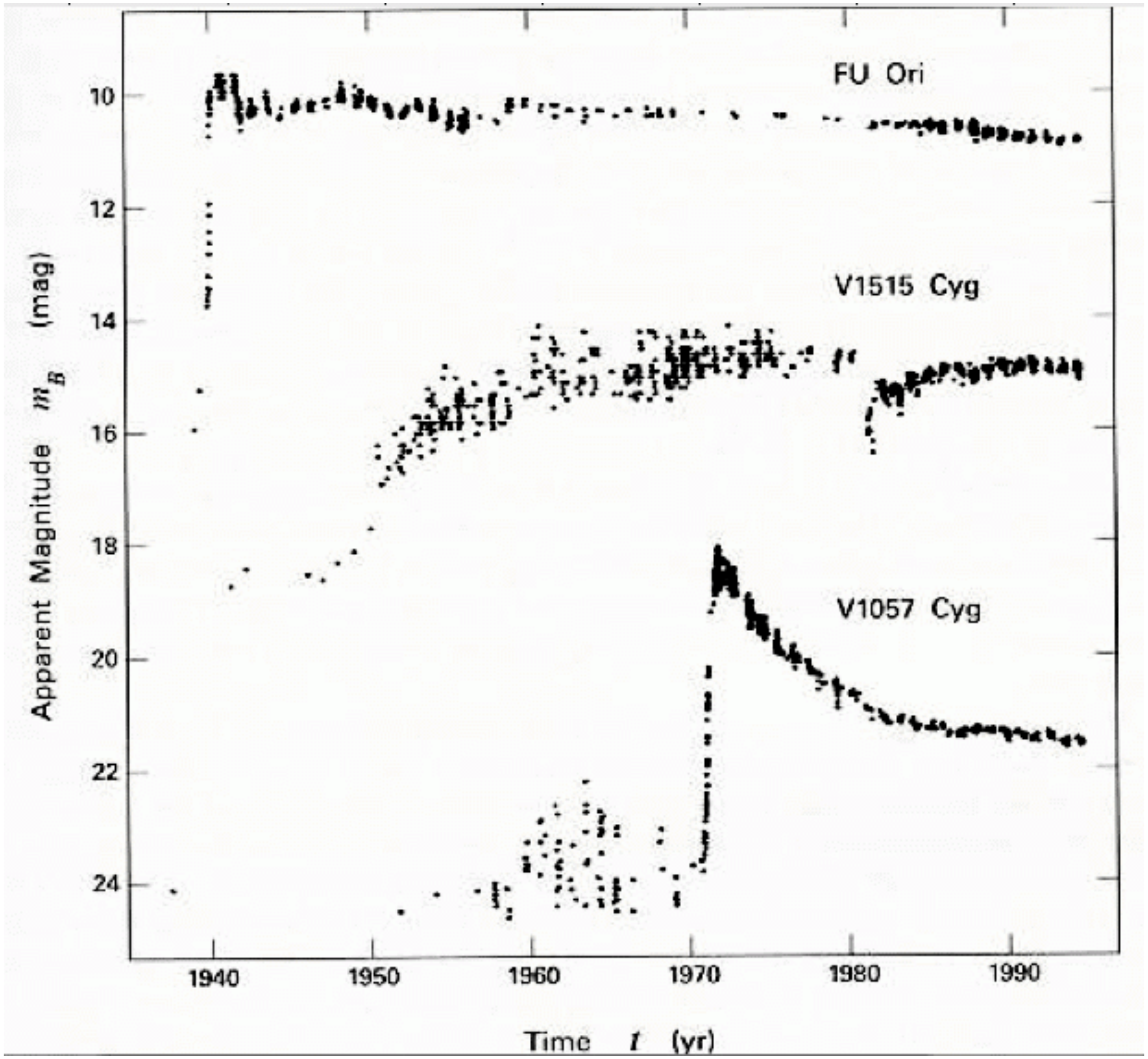
Unstable disk: spiral waves, angular momentum transport, strong accretion!!

Gravitational (in)stability



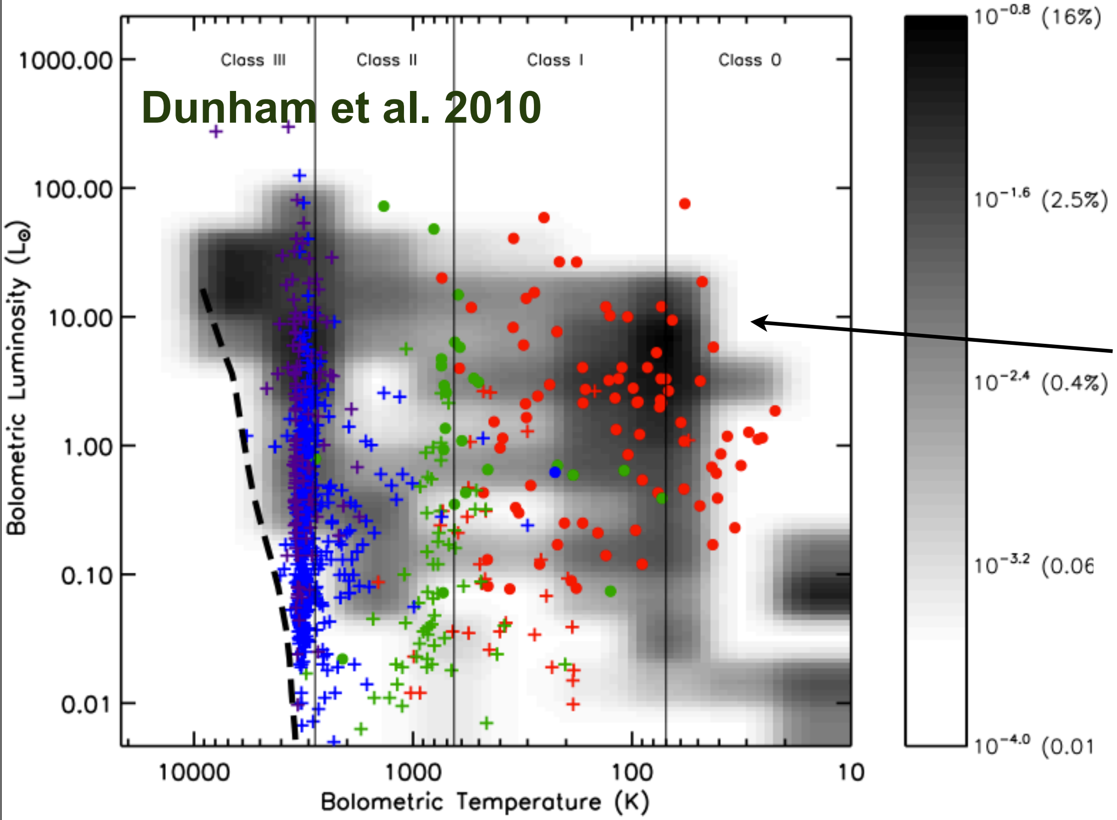
Rice & Armitage

FU Orionis stars

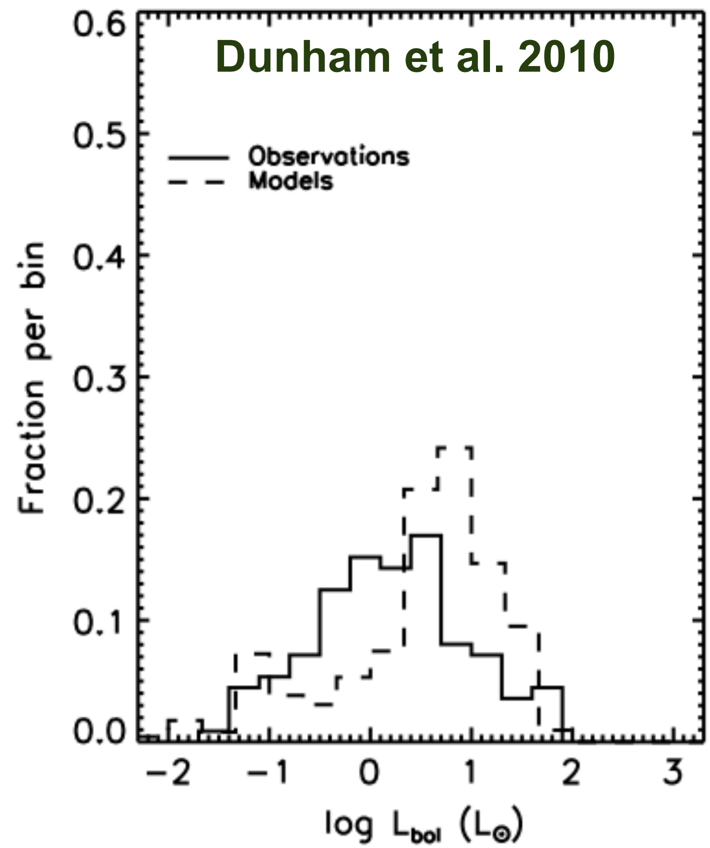


McNeal's Nebula: a new FU Ori?

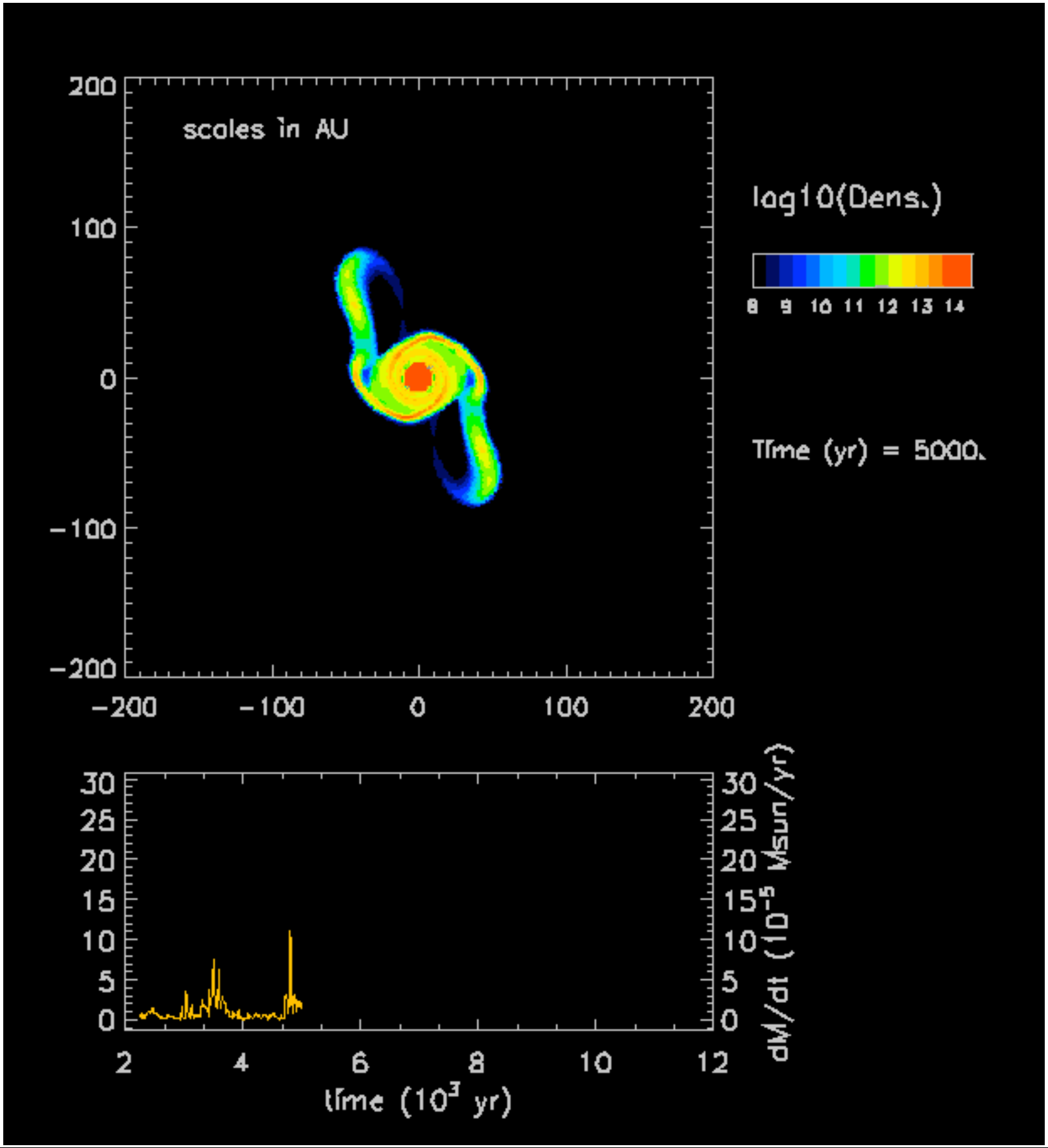
The Luminosity Problem



Grey gives predicted luminosites from models



This is a longstanding problem: the required accretion luminosities are on average higher than the observed luminosities.



Voroboyov & Basu

Summary

- In Keplerian orbits, the inner disk is rotating faster than the outer disk
- Viscosity acts as a friction. Mass is transported inward, angular momentum outward.
- This causes mass to accrete onto the central source (Will Fischer's lecture tuesday), and the disk to spread outward.
- Viscosity can be do to thermal motions in a gas (not important for disks), magneto rotational instability, or spiral waves created by gravity.
- Young stars may go through episodic accretion: gas first accumulates onto a disk until it undergoes gravitational instability, then the mass is dumped onto the disk in a major event.