Wind Blown Bubble Solution: From Shu et al. 1990

We start by assuming a Singular Isothermal Sphere with an additional azimuthal dependence on density given by the function $Q(\mu)$ (where $\mu = cos(\theta)$):

$$\rho(r,\theta) = \frac{c_s^2}{2\pi G r^2} Q(\mu) \tag{1}$$

where

$$\int_0^1 Q(\mu) d\mu = 1 \tag{2}$$

We could use $Q(\mu)$ to describe a more flattened core with the mass concentrated near the "equator" ($\mu = 0$).

We also assume the standard mass infall rate and assume a constant fraction of the infalling gas is ejected in an outflow, giving a total mass outflow rate of M_w.

$$\dot{M} \approx \frac{c_s^3}{G}, \;\; \dot{M}_w = f\dot{M} \tag{3}$$

We can then write the momentum per steradian emitted by the wind as.

$$P(r,\theta) = \frac{\dot{M}_w v_w}{4\pi} P(\mu) \tag{4}$$

We have given the wind an azimuthal dependence described by the equation $P(\mu)$ where

$$\int_0^1 P(\mu) d\mu = 1 \tag{5}$$

We can use $P(\mu)$ to create an outflow which is concentrated along the "pole" ($\mu = 1$). Assuming momentum conservation but not energy conservation (i.e. a snowplow), the mass per steradian, M_{sr}, grows

$$\frac{dM_{sr}}{dt} = \frac{c_s^2}{2\pi G} Q(\mu) v_s \tag{6}$$

and the momentum per steradian grows by:

$$\frac{d}{dt}(M_{sr}v_s) = \frac{\dot{M}_w v_w}{4\pi} P(\mu) \tag{7}$$

Now we balance momentum change per steradian due to wind with ram pressure per steradian ($v_s dM_{sr}/dt$)

$$v_s \frac{dM_{sr}}{dt} = \frac{c_s^2}{2\pi G} Q(\mu) v_s^2 = \frac{\dot{M}_w v_w}{4\pi} P(\mu) \tag{8}$$

The resulting solution gives a constant velocity for a given value of μ:

$$v_s = \left(\frac{\dot{M}_w}{2\dot{M}}\right)^{1/2} (c_s v_w)^{1/2} \beta(\mu),\ where\ \beta(\mu) = \left(\frac{P(\mu)}{Q(\mu)}\right)^{1/2} \tag{9}$$

The radius as a function of μ is given by:

$$r_s = (f/2)^{1/2} (c_s v_w)^{1/2} t \beta(\mu) \tag{10}$$

This gives a "Hubble" law for outflows:

$$v_s = \frac{r_s}{t} \tag{11}$$

In reality, the shape of the bubble is given by $\beta(\mu)$, which in reality is just the ratio of two fudge factors.