Lectures 18 and 19

Optical Depth vs. Density

Imaging a sphere of mass M, constant density ρ, radius R, and absorption by mass of κ_λ. The the optical depth is

$$\tau = \kappa_\lambda \rho R = \kappa_\lambda \frac{3M}{4\pi R^3} R = \kappa_\lambda \frac{3M}{4\pi R^2} \tag{1}$$

Thus as the radius decreases and the density increases then the core eventually becomes optically thick. In reality, this will be the center of a collapsing cloud and the density will not be constant; however, the basic argument remains. As we pack more and more matter into a small space, the optical depth increases.

Opacity Limited Fragmentation

We see that as a cloud collapses, the optical depth increases. At the same time, the Jeans length and mass will decrease. This leads to two competing effects. On one hand, the density increases, leading to higher optical depths. At the same time, Fred Hoyle in the 1950s predicted that as a cloud collapsed, it would continue to fragment into smaller and smaller pieces as the density increased. The assumption was that the temperature was constant, a condition needed for Jeans fragmentation. Will this lead to an increase or decrease in the optical depth?

We remember from previous lectures that the Jeans mass is given by $M_J = \rho \lambda_J^3$, where λ_J is the Jeans length:

$$M_J = \left(\frac{\pi k}{G m_H}\right)^{3/2} T^{3/2} \rho^{-1/2} \tag{2}$$

or

$$M_J = \pi^{3/2} c_s^3 (G^3 \rho)^{-1/2} \tag{3}$$

To analyze the optical depth, let's consider the Jeans length,which is given by:

$$\lambda_J = \pi^{1/2} c_s (G\rho)^{-1/2} \tag{4}$$

Then the optical depth is:

$$\tau = \kappa_\lambda \rho \lambda_J = \chi_\lambda \pi^{1/2} c_s (\rho/G)^{1/2} \tag{5}$$

so that τ increases as the cloud collapses and ρ increases. Eventually, collapse and Hoyle fragmentation will produce optically thick fragments. At this point, the cores are no longer isothermal. Since Jeans fragmentation requires an isothermal gas, this will also halt the fragmentation of the collapsing gas into progressively smaller pieces. Low and Lynden-Bell (1976) calculated that this occurs at a few hundredths of a stellar mass. This suggests the initial pieces are much smaller than a stellar mass, they will need to grow by accreting gas from the surrounding cloud.

Pre-main sequence evolution: Following analysis of Hartmann, Chapter 11

We start with the equation for an adiabatic gas:

$$P\rho^{-\gamma} = K_1 = P\rho^{-(1+1/n)} \tag{6}$$

where $\gamma = 5/3$ and n, the polytropic index is given by $n = 3/2$. This would be the case for a fully convective star, where convection maintains an adiabatic equation of state from the center of the star to the photosphere. As we did last semester, we can estimate the central pressure in the star by the equation of hydrostatic equilibrium:

$$\frac{dP}{dR} = \frac{P_c}{R} = -\rho \frac{GM}{R^2} \tag{7}$$

which for a constant density star ($\rho = M_\star / 4\pi R_\star^3$).

$$P_c = \rho \frac{GM}{R} = \frac{3GM_\star^2}{4\pi R_\star^4} \propto \frac{GM_\star^2}{R_\star^4} \tag{8}$$

Using ideal gas law, $P\rho^{-\gamma} = K$ and $P = \rho kT/\mu m_H$ we get

$$T_c \propto \frac{GM_\star}{R_\star} \tag{9}$$

To solve for the luminosity and effective temperature of the star, we must consider the stellar atmosphere where convection no longer is occuring. Here again we use hydrostatic equilibrium, approximating the atmosphere as a plane-parallel atmosphere with a gravitational constant $g = GM/R^2$.

The resulting equation is:

$$\frac{dP}{dz} = -\rho g \tag{10}$$

We integrate downward in the stellar atmosphere until $\tau = \int \kappa \rho dz = 2/3$. Accordingly:

$$P_{eff} = \frac{2}{3} \frac{g}{\kappa_{rm}} \tag{11}$$

where

$$\frac{1}{\kappa_{rm}} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu(T)}{dT} d\nu}{\int_0^\infty \frac{dB_\nu(T)}{dT} d\nu} \tag{12}$$

Adopt an opacity law for the stellar atmosphere, $\kappa_{rm} = \kappa_0 \rho T_{eff}^a$. Due to H^- opacity, κ_{rm} is a strong function of temperature with $a \sim 10$. For a constant density star:

$$P_{eff} = \frac{2}{3} \frac{GM_\star}{R_\star^2 \kappa_0 \rho T_{eff}^a} = \frac{2}{3} \frac{G\rho 4\pi R_\star^3}{R_\star^2 3 \kappa_0 \rho T_{eff}^a} = \frac{8}{9} \frac{R_\star}{\kappa_0 T_{eff}^a} \tag{13}$$

For an adiabatic star (remember that $PT^{-5/2} = K_2$):

$$\frac{T_{eff}}{T_c} = \left(\frac{P_{eff}}{P_c} \right)^{-2/5} \tag{14}$$

By substituting the relationships for P_c, T_c, and P_{eff}, we get

$$T_{eff} \propto R^{2.5/(2.5+a)} M^{0.5/(2.5+a)} \tag{15}$$

and

$$L_{eff} \propto R^{(15+2a)/(2.5+a)} M^{2/(2.5+a)} \tag{16}$$

If a is large, then there T_{eff} is relatively constant while L_{eff} decreases as the radius decreases.

Convective (Hayashi) Tracks: Following Hartmann, Chapter 11

From the virial theorem we find that $2U + W = 0$ where U is total kinetic energy and W is total potential energy. Thus $U = W/2$ and the total energy $-W/2$.

For $\gamma = 5/3$ ($P \propto \rho^\gamma$):

$$L_\star = -\frac{d}{dt}\frac{3}{7}\frac{GM_\star^2}{R_\star} \tag{17}$$

with a high value of a, for a given mass

$$T_{eff} = \left(\frac{L_\star}{4\pi\sigma R_\star^2}\right)^{1/4} \approx constant \tag{18}$$

Using Teff as a constant and using $L_\star = 4\pi R_\star^2 T_{eff}^4$, we get:

$$R_{star}^2 = \frac{3GM_\star^2}{28\pi\sigma R_\star^2 T_{eff}^4}\frac{dR}{dt} \tag{19}$$

Integrating both sides (and starting with $t = 0$ and R being very large):

$$t = \frac{GM_{star}^2}{28\pi\sigma T_{eff}^4 R_\star^3} \tag{20}$$

$$4\pi R_\star^2 T_{eff}^4 = L_\star = (4\pi\sigma T_{eff}^4)^{1/3}\left(\frac{GM_\star^2}{7t}\right)^{2/3} \tag{21}$$

Defining the Kelvin-Helmoltz as

$$t_{KH} = \frac{3}{7}\frac{GM_\star^2}{R_0} \text{ and } L_0 = 4\pi\sigma T_{eff}^4 R_0^2 \tag{22}$$

where R_0 is the radius when $t = t_{KH}/3$, we get our final answer:

$$L_\star = 4\pi\sigma T_{eff}^4 R_0^2\left(\frac{GM_\star^2}{7R_0 t}\right)^{2/3} = L_0\left(\frac{3t}{t_{KH}}\right)^{-2/3} \tag{23}$$

Radiative (Henyey) Tracks: Following Hartmann, Chapter 11

Convection will when the radiative gradient is to high. This gradient is given by:

$$\nabla R = \frac{dlnT}{dlnP} = \frac{3\kappa_R P}{64G\pi\sigma T^4}\frac{L}{M} \tag{24}$$

if $\nabla R < 0.4$ (remember that $R = 4$ for $\gamma = 5/3$ and $P \propto T^{5/2}$) then convection stop. This will occur as the star contracts and the luminosity drops (P_c/T_c^4 does not decrease with radius). At this point, radiation begins to be the main means for transporting energy, and we can use the radiative diffusion equation to estimate the luminosity:

$$L = \frac{64\pi\sigma r^2 T^3}{3\rho\kappa_r}\frac{dT}{dr} \tag{25}$$

using the basic equations for stellar structure and the Kramer's opacity

$$T \propto \frac{M_\star}{R_{star}},\ \text{and}\ \rho \propto M_\star/R_\star^3,\ \text{and}\ \kappa \propto \rho T^{-3.5} \tag{26}$$

and by assuming that $dT/dR = T_c/R_\star$, we get:

$$L_\star \propto M_\star^{5.5} R_\star^{-0.5} \tag{27}$$

The Stellar Birthline: Following Hartmann, Cassen & Kenyon (1997)

What we want to do is calculate the evolution of a protostar as it accretes gas. We focus on the evolution of mass and radius instead of the temperature and luminosity. One reason is that much of the luminosity of the star is accretion luminosity, and not emission from a stellar photosphere as in a normal star. Another reason is that accreting layers of gas may absorb the stellar radiation. The end result is that the observed luminosity may be very different than the luminosity of the central protostar. The final reason is that we want to calculate the initial conditions for pre-main sequence contraction; i.e. we want to find the initial radius for a given stellar mass.

Stahler (1983) introduced the idea of a stellar birthline. The birthline is the locus in the HR diagram where we expect young pre-main sequence stars to appear after the protostellar phase, i.e. the luminosity and temperature of a pre-main sequence star of mass M at its initial radius R. To get the stellar birthline, we calculate the mass and radius of a protostar as it evolves in time. Then, when accretion shuts off, we simply match the mass and radius to that of a pre-main sequence track to get the luminosity and temperature and put the star on the HR diagram. The resulting curve on the HR diagram is the birthline.

Imagine an accreting a star. As the gas falls onto the central star, it lowers the potential energy onto the star by:

$$\Delta W = -\frac{GM_\star \Delta m}{R} \tag{28}$$

At the same time, the gas may bring internal energy:

$$\Delta U = \epsilon \frac{GM_\star \Delta m}{R} \tag{29}$$

If gas fell from infinity and then all the potential energy turned to internal energy, than $\epsilon = 1$ and there is no net energy gain. If the gas started at a bound orbit, the $\epsilon < 1$. If the gas is totally cold when it lands on the star, $\epsilon = 0$.

For a pre-main sequence star:

$$L_\star = -\frac{3}{7}\frac{G(M)^2}{R}\left(\frac{\dot{R}}{R}\right) \tag{30}$$

For a protostar we write this as a balance.

$$-\frac{3}{7}\frac{G(M)^2}{R}\left(\frac{\dot{R}}{R}\right) = \dot{M}\frac{GM}{R}(\epsilon - 1) - \frac{dE_{surf}}{dt} + L_D \tag{31}$$

The term dE_{surf}/dt is the rate of radiative energy loss over the stellar surface. L_D is the luminosity from Deuterium burning. On the stellar surface, imagine that the accreting gas lands on a fraction of the surface given by the filling factor δ. Then:

$$\frac{dE_{surf}}{dt} = 4\pi R^2\left[\delta F_{acc} + (1-\delta)\sigma T_{eff}^4\right] \tag{32}$$

where F_{acc} is the flux radiating from regions of accretion. The luminosity of the protostar is then:

$$L_{phot} = 4\pi R^2(1-\delta)\sigma T_{eff}^4 \tag{33}$$

which will go to the usual $4\pi R^2\sigma T_{eff}^4$ when $\delta \rightarrow 0$. Let us define α, where

$$\dot{M}\frac{GM}{R}\alpha \equiv \dot{M}\frac{GM}{R}\epsilon - 4\pi R^2\delta F_{acc} \tag{34}$$

where $4\pi R^2\delta F_{acc}$ is the energy radiated away by the accretion shock. Thus, if α is close to 1, the infalling gas keeps all of its energy. If $\alpha = 0$, it loses that energy - perhaps by radiating it back into space. Now, we can write the luminosity as:

$$L_\star = -\frac{3}{7}\frac{GM_\star^2}{R_\star}\left[\left(\frac{1}{3} - \frac{7\alpha}{3}\right)\frac{\dot{M}}{M_\star} + \frac{\dot{R}_\star}{R_\star}\right] + L_D \tag{35}$$

Let us consider various limits of the equation. If $\dot{M} = 0$ (no accretion) and $L_D = 0$ (no Deuterium burning), then:

$$L_\star = -\frac{3}{7}\frac{GM_\star^2}{R_\star}\frac{\dot{R}_\star}{R_\star} \tag{36}$$

On the other hand, if $L_\star = 0$, i.e. the luminosity is totally absorbed by the infalling gas, and $\alpha = 0$ (infalling gas is cold) and $L_D = 0$ (no Deuterium burning) then:

$$-\frac{1}{3}\frac{\dot{M}}{M_\star} = \frac{\dot{R}_\star}{R_\star} \tag{37}$$

or

$$-\frac{1}{3}\frac{dlnM_\star}{dt} = \frac{dlnR_\star}{dt} \tag{38}$$

In this case $R \propto M^{-1/3}$. On the otherhand, imagine that $\alpha = 1$. In this limit:

$$2\frac{\dot{M}}{M_\star} = \frac{\dot{R}_\star}{R_\star} \tag{39}$$

and $R \propto M^2$. So in the limit of no Deuterium burning, and the luminosity of the star is smothered by accretion, α controls whether the star grows or shrinks. If $\alpha > 1/7$, the added gas is hot and the star grows, if $\alpha < 1/7$, the added gas is cold and the star shrinks.

What is α? Consider the material that is accreting onto the central star has a temperature T_{acc}.

$$T_{acc}\Delta m = \frac{\Delta U}{C_V} = \frac{2}{3}\frac{\mu m_H}{k}\Delta U, \text{ or } T_{acc} = \frac{2}{3}\frac{\mu m_H}{k}\frac{\Delta U}{\Delta m} \tag{40}$$

The internal energy divided by $C_V = (3/2)(\mu m_H/k)$ for a $n = 3/2$ polytropic star (i.e. star convective star with $\gamma = 5/3$):

$$M < T > \equiv \int T dM = \int \frac{u}{C_V} dM = \frac{3}{7}\frac{GM_\star^2}{R_\star}\left(\frac{2}{3}\frac{\mu m_H}{k}\right) = \frac{3}{7}\frac{\mu m_H}{k}\frac{GM_\star^2}{R_\star} \tag{41}$$

where u is the specific internal energy (internal energy per gram). Now we can write

$$\frac{\Delta U}{\Delta m} = \alpha\frac{GM}{R} \tag{42}$$

and solving for α:

$$\alpha = \left(\frac{\Delta U}{\Delta m}\right)\frac{R}{GM_\star} \tag{43}$$

we can further write:

$$\alpha = \frac{9}{14} \frac{T_{acc}}{< T >} \tag{44}$$

Thus if $T_{acc} = (2/9) < T >$ the $\dot{R}/R \sim 0$ if $L_D \sim 0$. For T_{acc} greater than this value, the star will grow, for T_{acc} less than this value, the star will shrink.

Deuterium Burning

Before the onset of Hydrogen burning, Deuterium can be fused with Hydrogen at a temperature of a 1×10^6 K. The reaction for Deuterium burning is:

$$^2H + {}^1H \rightarrow {}^3He + \gamma \tag{45}$$

The resulting equation produces $\Delta E_D \equiv 5.5$ MeV. The onset of Deuterium burning is at 1×10^6 K, as opposed to 1×10^7 K for the proton-proton chain. The energy generation is

$$\epsilon_D = \left[\frac{D}{H}\right] \epsilon_0 \left(\frac{\rho}{1\ g\ cm^{-3}}\right) \left(\frac{T}{1 \times 10^6\ K}\right)^{n_D} \tag{46}$$

where $n_D = 11.8$ and $\epsilon_0 = 4.19 \times 10^7$ erg g^{-1} s^{-1}.

The luminosity for Deuterium burning was calculated by Stahler (1988) for a convective star ($n = 3/2$ polytrope)

$$L_D = 1.92 \times 10^{17} f \left[\frac{D}{H}\right] \left(\frac{M}{M_\odot}\right)^{13.8} \left(\frac{R}{R_\odot}\right)^{-14.8} L_\odot \tag{47}$$

where [D/H] is the interstellar Deuterium abundance (2.5×10^{-5}) and f is the fraction of Deuterium left relative to the interstellar abundance. During protostellar evolution, the amount of Deuterium is quickly used up, unless more is brought in. The change in the mass of Deuterium is determined by the equation:

$$\frac{dfM}{dt} = \dot{M} - \frac{L_D}{\beta_D} \tag{48}$$

where β_D is the energy available per gram of gas with a typical [D/H] ratio (9.2×10^{13} ergs gm^{-1}). If $L_D = \dot{M}\beta_D$, then an equilibrium is achieved. A higher luminosity will deplete the Deuterium.