Properties of a constant density star

In the fall semester, we made use of the fact that "stars do not twinkle", i.e. stars are self gravitating, thermally supported, massive spheres of gas which are in hydrostatic equilibrium. We further, for the sake of developing intuition, determined the properties of a star with the assumption of constant density. The resulting **approximate** equations are the following:

Constant volume density:

$$\rho = \frac{3M}{4\pi R^3} \tag{1}$$

Central pressure

$$P_c = \frac{3}{8\pi} G \frac{M^2}{R^4} \tag{2}$$

Central temperature

$$T_c = \frac{\mu m_H}{2K} G \frac{M}{R} \tag{3}$$

Luminosity

$$L = \frac{16\pi^2 G\sigma}{9\chi\rho} \left(\frac{\mu m_H}{k}\right)^4 M^3 \tag{4}$$

Pre-Main Sequence Evolution

In the pre-main sequence phase, the star is powered by the liberation of gravitational potential energy through the slow, quasistatic contraction of the star. This can be seen by using the constant density approximation.

The gravitational potential energy for a constant density star is:

$$U = -\frac{3}{5}\frac{GM^2}{R} \tag{5}$$

Then assuming virial equilibrium and assuming thermal gas pressure:

$$W = \frac{1}{2}U \tag{6}$$

The luminosity can be then be related to the change in radius:

$$L = \frac{dW}{dt} = -\frac{3}{10}\frac{GM^2}{R^2}\frac{dR}{dt} \tag{7}$$

We can also plot pre-main sequence evolution in terms of T_c vs P_c

$$T = 4.1 \times 10^6 \mu \left(\frac{M}{M_\odot}\right)^{\frac{2}{3}} \rho^{-\frac{1}{3}} \tag{8}$$

For stars with masses in excess of 5.5 $M_\odot$ where radiation pressure dominates.

$$T = 1.92 \times 10^7 \left(\frac{M}{M_\odot}\right)^{\frac{1}{6}} \rho^{-\frac{1}{3}} \tag{9}$$

As the star contracts, the density and temperature rises until the onset of nuclear fusion at central temperatures of 10^7 K. At this point, the star becomes a main sequence star.

There are three major timescales in stellar evolution:

The **free fall time** is the time for which a gas cloud collapse onto a protostars (this is really the lower limit, since it assumes no pressure, just gravity):

$$t_{ff} = \left(\frac{3\pi}{32G\rho}\right)^{\frac{1}{2}} \tag{10}$$

For a density of 10^4 H_2 molecules cm^{-3} and assuming a 9% He/H ratio by number, we get a volume density of $4.5x10^{-20}$ cm^{-3} and a free fall time of 300,000 years.

The **Kelvin-Helmholtz** time is the duration over which a star can produce luminosity through quasistatic contraction (i.e. the pre-main sequence phase).

$$t_{KH} = \frac{GM^2}{RL} \tag{11}$$

For a M = 1 $M_\odot$, R=2 $R_\odot$ and, L = 5 $L_\odot$, $t_{KH} = 30 \times 10^6$ years.

The **nuclear time** is the duration over which a star contain sustain itself by fusion of Hydrogen atoms. This is given by:

$$t_{nuc} = 10^{-3}\frac{Mc^2}{L} \tag{12}$$

For a star with M = 1 $M_\odot$ and L = 1 $L_\odot$, then $t_{nuc} = 10^7$ years.

Free-Fall timescale: collapse time for a molecular cloud.

We begin by assuming a constant density sphere with an initial density ρ_0 and initial radius of r_0. We also assume that initially the sphere is at rest. If we further assume that there is not pressure, Newton's Laws give us:

$$\frac{d^2r}{dt^2} = -\frac{GM}{R^2} = -\frac{G4\pi\rho_0 r_0^3}{3r^2} \tag{13}$$

To solve this equation, we multiply each side by dr/dt and integrate.

$$\frac{dr}{dt}\frac{d^2r}{dt^2} = -\frac{G4\pi\rho_0 r_0^3}{3r^2}\frac{dr}{dt} \tag{14}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{G8\pi\rho_0 r_0^3}{3r} \tag{15}$$

$$\left(\frac{d(r/r_0)}{dt}\right)^2 = \frac{G8\pi\rho_0}{3}\left(\frac{r_0}{r} - 1\right) \tag{16}$$

$$\frac{d(r/r_0)}{dt} = -\left(\frac{G8\pi\rho_0}{3}\right)^{\frac{1}{2}}\left(\frac{r_0}{r} - 1\right)^{\frac{1}{2}} \tag{17}$$

Note that we have adopted the root of the equation where dr/dt will be negative, which is what we expect. We now substitute $r/r_0 = cos^2(\beta)$ and then integrate by β on the left size and t on the right size.

$$-2cos(\beta)sin(\beta)\frac{d\beta}{dt} = -\left(\frac{G8\pi\rho_0}{3}\right)^{\frac{1}{2}}\frac{sin(\beta)}{cos(\beta)} \tag{18}$$

Using standard trig identities we find:

$$2cos^2(\beta)d\beta = \left(\frac{G8\pi\rho_0}{3}\right)^{\frac{1}{2}} dt \tag{19}$$

$$1 + cos(2\beta)d\beta = \left(\frac{G8\pi\rho_0}{3}\right)^{\frac{1}{2}} dt \tag{20}$$

Integrating these from t = 0 to t and $\beta = 0$ to β we get:

$$\beta + \frac{1}{2}sin(2\beta) = \left(\frac{G8\pi\rho_0}{3}\right)^{\frac{1}{2}} t \qquad (21)$$

Now during the collapse, r goes from r_0 to 0 and β accordingly goes from 0 to $\pi/2$. Thus, we solve for t when $\beta = \pi/2$

$$\frac{\pi}{2} = \left(\frac{G8\pi\rho_0}{3}\right)^{\frac{1}{2}} t_{ff} \qquad (22)$$

which has the solution:

$$t_{ff} = \left(\frac{3\pi}{32G\rho_0}\right)^{\frac{1}{2}} \qquad (23)$$

A pecuilliar aspect of this solution is that gas at every initial radius collapses to the center at the same time.

Kelvin-Helmoltz timescale: duration over which energy can be derived from gravity

Total gravitational energy: $U \sim -\frac{GM^2}{R}$

(or $U = -\frac{3}{5}\frac{GM^2}{R}$ for a constant density sphere)

Kelvin-Helmholz time scale is thus

$$t_{KH} = \left|\frac{U}{L}\right| = \frac{GM^2}{RL} \qquad (24)$$

Nuclear Timescale: duration of Hydrogen burning

Energy produced per baryon by $P - P$ chain or CNO cycle: $\sim 6.5MeV$.

Mass of a nucleon (1 A.M.U.) : 930 MeV

Thus, about 0.7% of the mass per nucleon is converted to energy.

Since only the central $\sim$ 10% of the mass of a star is hot and dense enough for nuclear fusion, only about $\sim$ 0.1% of the stellar mass can be converted into energy.

Spectral Energy Distributions

Spectral energy distribution are composite spectra assembled from broad-band photometry and spectra with the goal of spanning a wavelength range covering several orders of magnitude.

Often plotted as $log(\lambda)$ vs λF_λ where F_λ is the flux density per wavelength (Energy per time per area per unit wavelength).

The Spectral Index is defined as

$$\alpha = \frac{dlog(\lambda F_\lambda)}{dlog(\lambda)} \tag{25}$$

Note that for a $alpha = 0$, i.e. a flat spectrum source,

For young stellar objects

- $\alpha > 0.3$ is a Class I source
- $-0.3 < \alpha < 0.3$ is a flat spectrum source
- $-3 < \alpha < -0.3$ is a Class II
- $\alpha = -3$ is a Class III object (pure photosphere)