

Lecture 21

Review: calculation of mean atomic weight of an ionized gas (μ)

Given a mass fraction X_i (or abundance) for an ionic (or atomic) species with atomic weight A_i , we can calculate μ by:

$$\frac{1}{\mu(ion)} = \sum \frac{X_i}{A_i} \quad (1)$$

For the electrons:

$$\frac{1}{\mu(e^-)} = \sum \frac{Z_i X_i}{A_i} \quad (2)$$

where Z_i is the charge of the ion. Finally, the total value of μ is:

$$\frac{1}{\mu} = \frac{1}{\mu(e^-)} + \frac{1}{\mu(ion)} = \sum \frac{Z_i X_i}{A_i} + \frac{X_i}{A_i} \quad (3)$$

Multiplying $1/\mu$ by ρ/m_H gives the number of particles needed for the ideal gas law. We use X for the mass fraction (abundance) of Hydrogen, Y for Helium, Z for everything else. We will consider an envelope, with normal "solar" abundances, and a core where Hydrogen has been depleted and the abundance is dominated by Helium. For ions:

$$\frac{1}{\mu(ion)} = X + \frac{1}{4}Y + \frac{1 - X - Y}{\langle A \rangle} \approx \frac{1}{4}(1 + 3X) \quad (4)$$

We now approximate $Y = 1 - X$ and $Z = 0$. Given $X = 0.707$ and $Y = 0.274$ and $\langle A \rangle \sim 20$ then $\mu(ion) = 1.29$. On the other hand, if $Y \sim 1$ then $\mu(ion) = 4$. For electrons:

$$\frac{1}{\mu(e^-)} = X + \frac{1}{2}Y + (1 - X - Y) \frac{\langle Z \rangle}{\langle A \rangle} \approx X + \frac{1}{2}Y = \frac{1}{2}(1 + X) \quad (5)$$

For a normal (envelope) abundance, $\mu(e^-) = 1.17$, for $X = 0$ (in the Hydrogen depleted core), $\mu(e^-) = 2$. The total value of μ is given by.

$$\frac{1}{\mu} = \frac{1}{\mu(e^-)} + \frac{1}{\mu(ion)} \approx \frac{3 + 5x}{4} \quad (6)$$

For an ionized gas with normal interstellar abundances, $\mu_{env} = 0.6$. If we convert the Hydrogen into Helium, then $\mu_{core} = 1.33$, *doubling the mean atomic weight per particle*.

The Effect of Hydrogen Burning

In the evolution of stars, the changing of the mean atomic weight of the gas particles by nuclear fusion can have a dramatic effect on the pressure, hence driving stellar evolution. Consider the ideal gas law:

$$P = \frac{\rho}{\mu m_H} kT \quad (7)$$

When we convert hydrogen into He, μ doubles, from 0.6 to 1.3. Thus, for the same temperature, the pressure drops by a factor of two.

The Main Sequence Phase

Is the main sequence phase characterized by constant energy? We can rewrite the Virial theorem, ignoring U_{rad} .

$$P_c = C \frac{M^2}{R^4} \quad (8)$$

The ideal gas law implies that

$$T_c = C \frac{\mu m_H}{k\rho} \frac{M^2}{R^4} \quad (9)$$

If we let $R = (3M/4\pi\rho)^{1/3}$ then:

$$T_c = C \left(\frac{4\pi}{3}\right)^{4/3} \frac{\mu m_H}{k} M^{2/3} \rho^{1/3} \quad (10)$$

From radiative diffusion

$$L = -\frac{64\pi\sigma r^2}{\kappa\rho} T^3 \frac{dT}{dr} \quad (11)$$

Approximating $dT/dr \sim T/R$, then:

$$L \propto \frac{RT^4}{\kappa\rho} \quad (12)$$

If we adopt the Kramers opacity $\kappa = \kappa_0 \rho T^{-3.5}$

$$L \propto \frac{M^{5.53} \rho^{0.166} \mu^{7.5}}{\kappa_0} \quad (13)$$

Since M is constant, and there is a low dependence on ρ (which is probably also close to constant):

$$\frac{L(t)}{L(0)} = \left[\frac{\mu(t)}{\mu(0)} \right]^{7.5} \quad (14)$$

The result is that as Hydrogen is converted into Helium, the Sun becomes more luminous. Early on, it was 25% less luminous. The faint young Sun paradox is that the early Earth was too cold for liquid water to exist. The most common explanation is that the greenhouse effect kept the surface warmer (the Earth's atmosphere was rich in CO_2 early on). As the same ages, it will get warmer. In about 1 billion years, predictions show the Oceans evaporating in a runaway greenhouse effect.

The Red Giant Phase

Now imagine that the central core of the star is depleted in Hydrogen, and that fusion is taking place primarily in a shell surrounding the central core (either by the P-P chain or the CNO cycle). What happens to the central core?

Let us consider the virial theorem for a shell. Now, we have a virial theorem similar to that which we considered for clouds and cores with an internal and external pressure (see lecture 4 equations, section 2).

$$\int_0^{V_c} P dV = P_s V_c + \frac{1}{3} \alpha \frac{GM_c^2}{R_c} \quad (15)$$

Now assume an isothermal gas where $P = \rho kT / \mu m_H$. Dividing by the volume, we get

$$P_s = \frac{3}{4\pi} \frac{kT}{\mu m_H} \frac{M_c}{R_c^3} - \frac{\alpha G}{4\pi} \frac{M_c^2}{R_c^4} \quad (16)$$

If we set $P_s = 0$, we get a minimum radius:

$$R_0 = \alpha \frac{\mu m_H}{3k} \frac{M_c \mu_c}{T_c} \quad (17)$$

Below this radius, there is no stable configuration. However, if there is external pressure, we can find a maximum value of P_s , by finding out where $dP_s/dR_c = 0$. At this point:

$$0 = -\frac{9}{4\pi} \frac{kT}{\mu m_H} \frac{M_c}{R_c^4} + \frac{\alpha G}{\pi} \frac{M_c^2}{R_c^5} \quad (18)$$

Giving the solution

$$R_1 = \frac{4\alpha G}{9R_\star} \frac{M_c \mu_c}{T_c} \quad (19)$$

Thus, if $R < R_0$, the core is unstable if the pressure was 0. If there is an external pressure, the core is unstable if $R < R_1$. It is stable for $R > R_1$. At R_1 , there is a maximum in the pressure where if the pressure is exceeded, the core will collapse. By substituting this back into the equation for P_s , we get

$$P_{s,max}(M_c) = C_1 \frac{T_c^4}{M_c^4 \mu_c^4} \quad (20)$$

The surrounding pressure is due to the weight of the envelope. We can estimate in by using the equation of hydrostatic equilibrium (assuming $M_c < M_\star$):

$$\frac{dP}{dm} = \int_0^{M_\star} \frac{Gm}{4\pi r^4} dm \quad (21)$$

Since $r < R$, where R is the radius of the star, we get the inequality:

$$P_{env} > \int_0^{M_\star} \frac{Gm}{4\pi R^4} dm = \frac{GM_\star^2}{8\pi R_\star^4} \quad (22)$$

Thus, the core is unstable if then

$$P_{s,max}(M_c) = C_1 \frac{T_c^4}{M_c^2} \mu_c^4 \geq \frac{GM_\star^2}{8\pi R_\star^4} \quad (23)$$

Now, we assume can use the equation

$$T_c = C_2 \frac{\mu_{env} m_H}{k} \frac{GM_\star}{R_\star} \quad (24)$$

giving the final relationship

$$\frac{M_c}{M_\star} \leq C_3 \left(\frac{\mu_{env}}{\mu_c} \right)^2 \quad (25)$$

Schönberg and Chandrasekhar derived that $C_3 = 0.37$.

What are μ_{env} and μ_c ? Let us assume that the envelope is dominated by Hydrogen, while the core is dominated by Helium. The total value of μ is given by:

$$\frac{1}{\mu} = \frac{1}{\mu(e^-)} + \frac{1}{\mu(ion)} \approx \frac{3 + 5x}{4} \quad (26)$$

This gives us $\mu_{env} = 0.6$ and $\mu_{core} = 1.33$

The critical ratio is when the mass of the Helium approaches the size:

$$\frac{M_c}{M_\star} \leq 0.37 \left(\frac{0.6}{1.33} \right)^2 \sim 0.1 \quad (27)$$

Thus, when 10% of the Hydrogen is converted into Helium, the star becomes unstable. At this point, it enters the Red Giant phase.