1. *Bound and Virialized Clouds*

To ascertain if a cloud is bound, we can first check to see if the kinetic energy of the cloud, T, is less than the gravitational potential energy. The kinetic energy of the cloud is given by:

$$T = \frac{1}{2} M_{cloud} < v^2 > \tag{1}$$

This can be related to the 1-D velocity dispersion by $v^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 3\sigma^2$ assuming that the velocity distribution is isotropic. The size vs linewidth relationship gives us $\sigma = kL^{0.5}$ which implies:

$$T = \frac{3}{2} M_{cloud} < \sigma^2 >= k_1 M L \tag{2}$$

and the density vs radius relationship gives us $M = kL^2$ or

$$T = k_2 M^2 / L = k_3 U \tag{3}$$

where U is the gravitational potential energy which is $\propto GM^2/R$. Thus, Larson's laws combine to suggest that T is proportional to U. Larson found evidence that the ratio of U to T implied stars are in virial equilibrium.

Does this imply that clouds are bound using more recent data? To test this, define a parameter of $\alpha = |U/T|$, where if $\alpha \geq 1$ then the cloud is bound and if $\alpha < 1$, the cloud is unbound. We assume that the star is a homogenous sphere, which is a lower limit to U:

$$U = -\frac{3}{5} \frac{GM^2}{R} \tag{4}$$

then the ratio is given by:

$$\alpha \geq \frac{2}{5} \frac{GM}{R\sigma^2} \tag{5}$$

Virial equilibrium implies $alpha = 2$ (potential energy is $-2T$.)

2. *Equations of Fluid Motion*

The equations for fluid motion describe the density ρ and velocity v of a continuous fluid. For the problem of star formation, we want to consider a compressible (i.e. density can change), self-gravitating fluid. In this case, the three basic equations for fluid motions are the following:

The continuity equation (conservation of mass):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{6}$$

where **v** is a vector.

The 2nd equation is the momentum equation, which basically states $F = ma$:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P - \rho \nabla \phi \tag{7}$$

Note that the equation for hydrostatic equilibrium can be derived from Euler's in the case equation when **v** is set to 0.

Finally, the gravitational potential ϕ is given by

$$\nabla^2 \phi = 4\pi G \rho \tag{8}$$

3. *The Jeans Length*

Consider an infinite, static medium with a density of ρ. Now, consider a small perturbation to this medium:

$$\mathbf{v} = \mathbf{v_0} + \delta\mathbf{v},\ \ \rho = \rho_0 + \delta\rho,\ \ \phi = \phi_0 + \delta\phi \tag{9}$$

If we subsititute these values into the continuity equation we get:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{10}$$

goes to

$$\frac{\partial\delta\rho}{\partial t} + \delta\mathbf{v}\cdot\nabla(\rho_0) + \rho_0\nabla\cdot\delta\mathbf{v} = 0 \tag{11}$$

where it is assumed $\mathbf{v_0} = 0$ and that all 2nd order terms, i.e. $\delta\rho\delta v$, go to 0.

For the momentum equation we get:

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P - \nabla\phi \tag{12}$$

goes to

$$\frac{\partial\delta\mathbf{v}}{\partial t} = -\frac{c_s^2}{\rho}\nabla\delta\rho - \nabla\delta\phi \tag{13}$$

where the $(\mathbf{v}\cdot\nabla)\mathbf{v}$ term is 2nd order and vanishes and c_s is the sound speed for an ideal, isothermal gas which is equal to

$$c_s = \sqrt{\frac{kT}{\mu m_H}} \tag{14}$$

(where $P = c_s^2\rho$ is the ideal gas law) and we have assumed that the unperturbed gas is in hydrostatic equilibrium, i.e.

$$\frac{c_s^2}{\rho_0}\nabla\rho_0 = -\nabla\phi_0 \tag{15}$$

Now we combine equations 6 and 8 by taking the derivative of equation 6 with respect to time:

$$\frac{\partial^2 \delta\rho}{\partial t^2} + \frac{\partial \delta\mathbf{v}}{\partial t} \cdot \nabla(\rho_0) + \rho_0 \nabla \cdot \frac{\partial \delta\mathbf{v}}{\partial t} = 0 \tag{16}$$

which if we assume ρ_0 is constant (and thus $\nabla\rho_0 = 0$) we get

$$\frac{\partial^2 \delta\rho}{\partial t^2} = -\rho_0 \nabla \cdot \frac{\partial \delta\mathbf{v}}{\partial t} \tag{17}$$

and then substituting equation 8 into the new equation:

$$\frac{\partial^2 \delta\rho}{\partial t^2} = c_s^2 \nabla^2 \delta\rho + \nabla^2 \delta\phi \tag{18}$$

Finally, using the Poisson equation, we arrive at our final equation:

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \rho_0 \left(\frac{c_s^2}{\rho_0} \nabla^2 \delta\rho + 4\pi G \delta\rho \right) \tag{19}$$

Note that the equation:

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \rho_0 \left(\frac{c_s^2}{\rho_0} \nabla^2 \delta\rho \right) \tag{20}$$

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \ c_s^2 \nabla^2 \delta\rho \tag{21}$$

is the wave equation for a pressure wave with sound speed c_s.

Let's rewrite this as 1-D equation (we can assume 1-D perturbations).

$$\frac{\partial^2 \delta\rho}{\partial t^2} = \rho_0 \left(\frac{c_s^2}{\rho_0} \frac{\partial^2 \delta\rho}{\partial x^2} + 4\pi G \delta\rho \right) \tag{22}$$

We adopt the following solution:

$$\delta\rho = \delta\rho_0 e^{i(wt-kx)} \tag{23}$$

If we plug this solution into the problem, the we then get the following dispersion equation:

$$w^2 = c_s^2 (k^2 - k_j^2) \tag{24}$$

where

$$k_j^2 = 4\pi G \frac{\rho_0}{c_s^2} \tag{25}$$

Thus, if $k > k_J$, w is real and we get a normal wave equation. However, if $k < k_J$, then the solution for w is imaginary. The resulting time dependence solution for the perturbation density is

$$\delta\rho = \frac{\delta\rho_0}{2} (e^{(|k^2-k_J^2|)^{1/2}t} + e^{-(|k^2-k_J^2|)^{1/2}t}) \tag{26}$$

which increases exponentially with time. In other words, gravity wins over pressure, and the pertubations are unstable. We can convert that into a length, the Jeans length

$$k_J = 2\pi/\lambda_J, \;\; \lambda_J = \sqrt{\frac{\pi c_s^2}{G\rho_0}} \tag{27}$$

the Jeans length can be turned into a mass

$$m_j = \rho_0 \lambda_J^3 = \frac{c_s^3 \pi^{3/2}}{G^{3/2} \rho_0^{1/2}} = \left(\frac{\pi k T}{\mu m_H G} \right)^{3/2} \rho_0^{-1/2} \tag{28}$$

4. Star Formation Efficiency

The star formation efficiency of a molecular cloud, ϵ, is defined as the following:

$$\epsilon = \frac{M_{stars}}{M_{stars} + M_{gas}} \tag{29}$$

where M_{stars} is the mass in stars and M_{gas} is the mass in molecular mass. It is essentially the fraction of mass that is converted into stars. Since it is difficult to determine the mass of individual stars, M_{gas} is calculated usually as the number of stars times the average stellar mass of 0.3 $M_{\odot}$, i.e. $M_{stars} = N_{stars} \times 0.5\ M_{\odot}$.