1. *Virial Theorem*

Last semester, we derived the Virial theorem from essentially considering a series of particles which attract each other through gravitation. The result was that

$$\frac{d^2I}{dt^2} = 2K + \Omega \tag{1}$$

where the Virial $I = \int r^2 dm$, Ω is the gravitational potential energy and K is the total kinetic energy. In the case of equilibrium where $\frac{d^2I}{dt^2} = 0$, we find that:

$$K = -\frac{1}{2}\Omega \tag{2}$$

which is very useful when considering stars in hydrostatic equilibrium. Unlike stars, for molecular clouds, there may be a significant external pressure. The virial theorem can be modified to take into account external pressure. In molecular cloud studies, a common way of writing the virial theorem is:

$$\frac{1}{2}\frac{d^2I}{dt^2} = 3\int_V PdV + \int_S P\ \mathbf{r}\cdot d\mathbf{S} + \Omega \tag{3}$$

where $3\int_V PdV$ is the integral of pressure over the volume of the cloud and is equal to $2K$, $\int_S P\ \mathbf{r}\cdot d\mathbf{S}$ is the integral of the external pressure over the surface of the cloud, and Ω once again the gravitational potential energy.

We define $K_s =\equiv \frac{1}{2}\int_S P\ \mathbf{r}\cdot d\mathbf{S}$. If the external pressure is a constant value P_0, then we can replace:

$$K_s = \frac{1}{2}P_0\int_S \mathbf{r}\cdot d\mathbf{S} = \frac{3}{2}P_0V \tag{4}$$

and we can write:

$$\frac{1}{2}\frac{d^2I}{dt^2} = 3\int_V (P - P_0)dV + \Omega \tag{5}$$

or if we assume a constant pressure

$$\frac{1}{2}\frac{d^2I}{dt^2} = 3(P - P_0)V_{cl} + \Omega \tag{6}$$

2. *The Stability of a Cloud (or Clump)*

Following Spitzer (1978), let's assume that a cloud is in equilibrium, i.e. $d^2I/dt^2 = 0$. The we rewrite equation 5 so that:

$$3P_0V = 3\int_V PdV + \Omega \tag{7}$$

assuming a homogenous sphere with sound speed c_s

$$4\pi P_0 R_{cl}^3 = 3c_s^2 M_{cl} - \frac{3}{5}\frac{GM_{cl}^2}{R_{cl}} \tag{8}$$

where c_s is the sound speed and the ideal gas law gives $P = c_s^2\rho$.

To examine the stability of the cloud to changes in external pressure, we solve the above equation for P_0 and then take the derivative with R

$$P_0 = \frac{1}{4\pi}\left(\frac{3c_s^2 M_{cl}}{R_{cl}^3} - \frac{3}{5}\frac{GM_{cl}^2}{R_{cl}^4}\right) \tag{9}$$

and

$$\frac{dP_0}{dR_{cl}} = \frac{1}{4\pi}\left(-\frac{9c_s^2 M_{cl}}{R_{cl}^4} + \frac{12}{5}\frac{GM_{cl}^2}{R_{cl}^5}\right) \tag{10}$$

Now let's consider what happens if we increase pressure. If the right side of the equation is negative, i.e.

$$R > R_{crit} = \frac{4}{15}\frac{GM_{cl}}{c_s^2} \tag{11}$$

then the increase in pressure will cause the cloud to shrink, but the value of P_0 required for equilibrium goes up as well. Eventually, the cloud comes into equilibrium, unless $R \leq R_{crit}$. At this point, $dP_0/dR_{cl} = 0$ and P_0 is at a maximum. Increasing P_0 will imply that there is no stable equilibrium and the cloud will collapse.

For a value of $M = 10^4$ M$_\odot$, $T_K = 20$ K (and consequently $c_s = 0.25$ km s^{-1}), $R_{crit} \sim$ 200 pc. This is larger than most clouds, molecular clouds cannot be supported by thermal pressure. We also note that $\alpha = |\Omega/K| = 6$, meaning that the cloud would be not in virial equilibrium.

3. *The Effect of the Magnetic Field*

Magnetic fields may have a stabilizing effect on clouds. Although the clouds are neutral, ionization by cosmic rays, and on the outer layers, UV radiation, may create a small number of electrons and ions. These would be coupled to the magnetic field, and in fact, freeze the magnetic field lines to the gas. Thus, the magnetic field pressure could exert a pressure.

$$\frac{1}{2}\frac{d^2I}{dt^2} = 3\int_V (P - P_0)dV + \Omega + M_B \tag{12}$$

where

$$M_B = \frac{1}{8}\pi \int (B^2 - B_0^2)dV \tag{13}$$

where B_0 is the external field.

$$M_B = \frac{1}{8\pi}\int (B^2)dV \equiv \frac{1}{6}\bar{B}^2 R_{cl}^3 = \frac{1}{6\pi^2}\frac{\phi^2}{R} \tag{14}$$

where b is order unity and $\bar{B}$ is the mean magnetic field.

$$\phi = \int 2\pi r B dr \equiv \pi R^2 \bar{B} \tag{15}$$

An important value is the ratio of M_{cl}/ϕ where the graviational potential energy is equal to the magentic energy term, $\Omega = -M_B$. For a uniform sphere, this occurs when:

$$\frac{3}{5}\frac{GM_{cl}^2}{R} = \frac{1}{6}\frac{\phi^2}{R} \tag{16}$$

or when

$$\frac{M_{cl}}{\phi} = \left(\frac{5}{18}\right)^{1/2}\frac{1}{\pi G^{1/2}} = \frac{1}{2\pi G^{1/2}} \tag{17}$$

The ratio M_{cl}/ϕ is known as the mass to flux ratio. If

$$\frac{M_{cl}}{\phi} > \frac{1}{2\pi G^{1/2}} \tag{18}$$

then gravity dominates over magnetic fields and the cloud is supercritical. Note, that since ϕ is conserved, the ratio does not change as the cloud collapse. If the ratio is < 1, then the cloud is sub-critical. Note that since

$$N(H_2) = \frac{M_{cl}}{\pi R_{cl}^2}, \;\; B = \frac{\phi}{\pi R_{cl}^2} \tag{19}$$

implies that

$$\frac{M_{cl}}{\phi} = \frac{N(H_2)}{B} \tag{20}$$

Measurements show that $\frac{N(H_2)}{B} \leq 1$, suggesting that magnetic fields do not support clouds against collapse.

4. *Turbulent "Pressure"*

We now consider the addition of turbulent energy to the kinetic energy term. If we just consider the thermal motions of the gas, the kinetic energy is given by:

$$K = \frac{3}{2} M_{cl} c_s^2 \tag{21}$$

where c_s is the sound speed. Remember that the total 3-D velocity dispersion is

$$\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 3c_s^2 \tag{22}$$

We can then define a pressure term:

$$P = \frac{3}{2} M_{cl} (c_s^2 + \sigma_{turb}^2) \tag{23}$$

where σ_{turb} is the 1-D turbulent dispersion. We can take this from the measured linewidth of the cloud, which is $\sigma_{tot}^2 = c_s^2 + \sigma_{turb}^2$.

$$P_{tot} = P_{th} + P_{turb} = \frac{3}{2} M_{cl} (\sigma_{tot}^2) \tag{24}$$

Then the equation becomes:

For a value of $M = 10^4$ $M_\odot$, $T_K = 20$ K (and consequently $c_s = 0.25$ km s^{-1}), $R_{crit} \sim$ 12 pc. This is similar to the size of most molecular clouds. Plus we know that $\alpha = |\Omega/K|$ is around 1, indicating that it is close to virial equilibrium.