

1. *Can dense cores be supported by thermal pressure? (a back of the envelope calculation)*

We begin by estimating the size of a thermally supported, cold (10 K), one solar mass globule of gas. The equation for hydrostatic equilibrium is

$$\frac{dP}{dr} = -\rho G \frac{M}{r^2} \quad (1)$$

or if we approximate $dP/dr = (P_c - P_0)/R$ where P_c is the central core pressure, P_0 is the outer pressure and R is the core radius, then:

$$P_c = -\rho G \frac{M}{r} \quad (2)$$

where we assume $P_0 \ll P_c$, and then by applying the ideal gas law ($P = c_s^2 \rho$):

$$c_s^2 \approx G \frac{M}{R} \approx \frac{kT}{\mu m_H} \quad (3)$$

Note, this is very similar to the virial equation. For a core with $M = 1M_\odot$ and $T = 10K$, we find $R = 0.15$ pc. This is very similar to the radii of molecular cores in low mass stars.

Would such cores be stable? We can examine the critical radius for a thermally supported core with an external pressure. We found from lecture 4 that this radius would be:

$$R > R_{crit} = \frac{4}{15} \frac{GM_{cl}}{c_s^2} \quad (4)$$

For the assumed temperature and mass above, $R > 0.04pc$. Thus, the cores would be stable.

What about Jeans instabilities? The Jeans mass is given by (Lecture 3):

$$m_j = \rho_0 \lambda_J^3 = \frac{c_s^3 \pi^{3/2}}{G^{3/2} \rho_0^{1/2}} = \left(\frac{\pi kT}{\mu m_H G} \right)^{3/2} \rho_0^{-1/2} \quad (5)$$

The resulting jeans mass would be around $4 M_\odot$. Thus, we have found that cores are in the regime that they could be considered to be stable condensations in hydrostatic equilibrium where the pressure is generated with thermal pressure.

2. *Hydrostatic Equilibrium and Bonner Ebert Spheres*

We have some experience with the equation for hydrostatic equilibrium for a spherically symmetric cloud of thermally supported gas: i.e. stars! The primary difference is that within a cloud we can assume the temperature is roughly constant, which saves us the trouble of solving for $T(r)$ using radiative transfer or convection.

We assume spherical symmetry. In this case, the term

$$\nabla P = dP/dr \tag{6}$$

where r is the radial position, and

$$\nabla^2 \phi = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) \tag{7}$$

We write the gravitational force as

$$\frac{d\phi}{dr} = -G \frac{M(r)}{r^2} \tag{8}$$

We now have three equations which describe our thermally supported gas ball. The equation for hydrostatic equilibrium becomes

$$\frac{1}{\rho(r)} \frac{dP(r)}{dr} = -\frac{d\phi}{dr} \tag{9}$$

where the equation of state is the ideal gas law

$$P = \rho c_s^2 \tag{10}$$

and the equation for the gravitational potential is:

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = 4\pi G \rho(r) \tag{11}$$

We can combine the first two equations to find:

$$\frac{c_s^2}{\rho(r)} \frac{d\rho(r)}{dr} = -\frac{d\phi}{dr} \tag{12}$$

$$\frac{d \ln \rho(r)}{dr} = -\frac{d}{dr} \left(\frac{\phi(r)}{c_s^2} \right) \quad (13)$$

We can integrate the equation for the gravitational potential

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = 4\pi G \rho(r) \quad (14)$$

to get

$$\int_0^r \frac{1}{r'^2} \frac{d}{dr'} \left(r'^2 \frac{d\phi}{dr'} \right) r'^2 dr' = \int_0^r 4\pi G \rho(r') r'^2 dr' \quad (15)$$

where

$$M(r) \equiv \int_0^r 4\pi G \rho(r') r'^2 dr' \quad (16)$$

Giving us

$$r^2 \frac{d\phi}{dr} = GM(r) \quad (17)$$

Now we just need to combine:

$$r^2 \frac{d\phi}{dr} = GM(r), \quad \frac{d \ln \rho(r)}{dr} = -\frac{d}{dr} \left(\frac{\phi(r)}{c_s^2} \right) \quad (18)$$

to get

$$\frac{d \ln \rho(r)}{dr} = -\frac{GM(r)}{c_s^2 r^2} \quad (19)$$

and

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \quad (20)$$

Let's go back to:

$$\frac{d \ln \rho(r)}{dr} = -\frac{d}{dr} \left(\frac{\phi(r)}{c_s^2} \right) \quad (21)$$

Let's assume $\phi(r=0) = 0$ and use the solution to the above equation of:

$$\rho(r) = \rho_c e^{-\phi(r)/c_s^2} \quad (22)$$

resulting in

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = 4\pi G \rho_c e^{-\phi(r)/c_s^2} \quad (23)$$

if we define

$$u \equiv \frac{\phi}{c_s^2} \quad (24)$$

$$\xi \equiv \left(\frac{4\pi G \rho_c}{c_s^2} \right)^{1/2} r \quad (25)$$

We arrive at the Lane-Emden equation

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{du}{d\xi} \right) = e^{-u} \quad (26)$$

To solve this equation, we set up the following boundary conditions

$$u(0) = 0 \quad (27)$$

$$\left. \frac{du(\xi)}{d\xi} \right|_{\xi=0} = 0 \quad (28)$$

and then we integrate outward in ξ .

Now we work on defining a dimensionless mass. We do this by considering the mass of the sphere:

$$M = \int_0^{r_0} 4\pi\rho(r)r^2 dr \quad (29)$$

now transform into our u and ξ using the transformations

$$\rho(r) = \rho_c e^{-\phi(r)/c_s^2} = \rho_c e^{-u}, \quad \xi \equiv \left(\frac{4\pi G \rho_c}{c_s^2} \right)^{1/2} r \quad (30)$$

and get:

$$M = 4\pi\rho_c \left(\frac{c_s^2}{4\pi G \rho_c} \right)^{3/2} \int_0^{\xi_0} e^{-u} \xi^2 d\xi \quad (31)$$

We then plug in the Lane-Emden equation to get

$$\int_0^{\xi_0} e^{-u} \xi^2 d\xi = \xi^2 \frac{du}{d\xi} \quad (32)$$

which gives the mass

$$M = 4\pi\rho_c \left(\frac{c_s^2}{4\pi G \rho_c} \right)^{3/2} \xi^2 \frac{du}{d\xi} \quad (33)$$

We now define the dimensionless mass as

$$m \equiv \frac{P_0^{1/2} G^{3/2} M}{c_s^4} \quad (34)$$

or for an outer boundary $\xi = \xi_0$:

$$m = \left(4\pi \frac{\rho_c}{\rho_0} \right)^{-1/2} \left(\xi^2 \frac{du}{d\xi} \right)_{\xi_0} \quad (35)$$

3. Critical Pressure

A property of a hydrostatically supported, isothermal sphere is a critical external pressure. We now have considered an external pressure for both a constant density sphere (using the virial theorem in lecture 4) and a Bonner Ebert sphere.

In lecture 3 we found a critical radius, where $R > R_{crit}$ for a stable cloud.

$$R_{crit} = \frac{4}{15} \frac{GM_{cl}}{c_s^2} \quad (36)$$

Using the equation derived from the Virial theorem

$$4\pi R_{cl}^3 P_0 = 3c_s^2 M_{cl} - \frac{3}{5} \frac{GM_{cl}^2}{R_{cl}} \quad (37)$$

we derive a critical external pressure

$$P_0 = 3.15 \frac{c_s^8}{G^3 M_{cl}^2} \quad (38)$$

For a Bonner Ebert sphere:

$$R_{crit} = 0.41 \frac{GM_{cl}}{c_s^2} \quad (39)$$

and

$$P_{crit} = 1.40 \frac{c_s^8}{G^3 M_{cl}^2} \quad (40)$$