Lecture 6: Modeling Molecular Cores as Bonnor-Ebert Spheres and Isothermal Spheres
Extinction map
Schmazl et al. 2010

Fig. 5.— Extinction map of the Filament L1495 with a resolution of 0.9′ derived from deep NIR observations with Omega2000. Contours are plotted in steps of $A_V = 5$ mag. We separate the filament into different subregions, which conform to Barnard’s Dark Objects (Barnard et al. 1927). The boxes indicate the positions of the zoom-ins shown in Fig.6.
Properties of Taurus Cores

Fig. 8.— Mass-size relation for our cores. The hatched area marks the region of incompleteness. The dotted lines represent the cases of constant volume density $M \propto r^3$ and column density $M \propto r^2$.

Fig. 9.— The Dense Core Mass Function (DCMF) of L1495--E, which was obtained by smoothing the core masses (indicated as vertical dashes along the abscissa) with a Gaussian kernel with FWHM = 0.3 dex. Points with error bars are plotted in separations of 50% of the FWHM. The dashed line shows the power law fit with slope $\Gamma = 1.2 \pm 0.2$ for $M > 2.0 M_\odot$. The hatched area indicates the region of incompleteness.

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Table 1: Properties of Dark Clouds, Clumps, and Cores

<table>
<thead>
<tr>
<th></th>
<th>Clouds (^a)</th>
<th>Clumps (^b)</th>
<th>Cores (^c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (M(_\odot))</td>
<td>10(^3)–10(^4)</td>
<td>50–500</td>
<td>0.5–5</td>
</tr>
<tr>
<td>Size (pc)</td>
<td>2–15</td>
<td>0.3–3</td>
<td>0.03–0.2</td>
</tr>
<tr>
<td>Mean density (cm(^{-3}))</td>
<td>50–500</td>
<td>10(^3)–10(^4)</td>
<td>10(^4)–10(^5)</td>
</tr>
<tr>
<td>Velocity extent (km s(^{-1}))</td>
<td>2–5</td>
<td>0.3–3</td>
<td>0.1–0.3</td>
</tr>
<tr>
<td>Crossing time (Myr)</td>
<td>2–4</td>
<td>(\approx) 1</td>
<td>0.5–1</td>
</tr>
<tr>
<td>Gas temperature (K)</td>
<td>(\approx) 10</td>
<td>10–20</td>
<td>8–12</td>
</tr>
<tr>
<td>Examples</td>
<td>Taurus, Oph, B213, L1709</td>
<td>L1544, L1498, Musca, B68</td>
<td></td>
</tr>
</tbody>
</table>

\(^a\) Cloud masses and sizes from the extinction maps by Cambrésy (1999), velocities and temperatures from individual cloud CO studies

\(^b\) Clump properties from Loren (1989) (\(^{13}\)CO data) and Williams, de Geus & Blitz (1994) (CO data)

\(^c\) Core properties from Jijina, Myers & Adams (1999), Caselli et al. (2002a), Motte, André & Neri (1998), and individual studies using NH\(_3\) and N\(_2\)H\(^+\)
Can Dense Cores be Supported by Thermal Pressure?

We begin by estimating the size of a thermally supported, cold (10 K), one solar mass globule of gas. The equation for hydrostatic equilibrium is

$$\frac{dP}{dr} = -\rho G \frac{M}{r^2}$$

(1)

or if we approximate $dP/dr = (P_c - P_0)/R$ where $P_c$ is the central core pressure, $P_0$ is the outer pressure and $R$ is the core radius, then:

$$P_c = -\rho G \frac{M}{r}$$

(2)

where we assume $P_0 \ll P_c$, and then by applying the ideal gas law ($P = c_s^2 \rho$):

$$c_s^2 \approx G \frac{M}{R} \approx \frac{kT}{\mu m_H}$$

(3)

Note, this is very similar to the virial equation. For a core with $M = 1M_\odot$ and $T = 10K$, we find $R = 0.15$ pc. This is very similar to the radii of molecular cores in low mass stars.
Are Dense Cores Supported by Thermal Pressure Stable?

Would such cores be stable? We can examine the critical radius for a thermally supported core with an external pressure. We found from lecture 4 that this radius would be:

\[ R > R_{\text{crit}} = \frac{4}{15} \frac{G M_{\text{cl}}}{c_s^2} \]  

(4)

For the assumed temperature and mass above, \( R > 0.04 \text{pc} \). Thus, the cores would be stable.

What about Jeans instabilities? The Jeans mass is given by (Lecture 3):

\[ m_j = \rho_0 \lambda_j^3 = \frac{c_s^3 \pi^{3/2}}{G^{3/2} \rho_0^{1/2}} = \left( \frac{\pi k T}{\mu m_H G} \right)^{3/2} \rho_0^{-1/2} \]  

(5)

The resulting jeans mass would be around 4 \( M_\odot \). Thus, we have found that cores are in the regime that they could be considered to be stable condensations in hydrostatic equilibrium where the pressure is generated with thermal pressure.
Solving for Hydrostatic Equilibrium in an Isothermal Gas: Bonnor-Ebert Spheres and Singular Isothermal Spheres
The Spherical Core (Cow?)

We assume spherical symmetry. In this case, the term

\[ \nabla P = \frac{dP}{dr} \quad (6) \]

where \( r \) is the radial position, and

\[ \nabla^2 \phi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) \quad (7) \]

We write the gravitational force as

\[ \frac{d\phi}{dr} = -G \frac{M(r)}{r^2} \quad (8) \]

We now have three equations which describe our thermally supported gas ball. The equation for hydrostatic equilibrium becomes
We now have three equations which describe our thermally supported gas ball. The equation for hydrostatic equilibrium becomes

$$\frac{1}{\rho(r)} \frac{dP(r)}{dr} = -\frac{d\phi}{dr}$$  \hspace{1cm} (9)$$

where the equation of state is the ideal gas law

$$P = \rho c_s^2$$  \hspace{1cm} (10)$$

and the equation for the gravitational potential is:

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = 4\pi G \rho(r)$$  \hspace{1cm} (11)$$

We can combine the first two equations to find:

$$\frac{c_s^2}{\rho(r)} \frac{d\rho(r)}{dr} = -\frac{d\phi}{dr}$$  \hspace{1cm} (12)$$

$$\frac{d}{dr} \left( \frac{\ln \rho(r)}{c_s^2} \right) = -\frac{d}{dr} \left( \frac{\phi(r)}{c_s^2} \right)$$  \hspace{1cm} (13)$$
We can integrate the equation for the gravitational potential

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = 4\pi G \rho(r)
\]  

(14)

to get

\[
\int_0^r \frac{1}{r'^2} \frac{d}{dr'} \left( r'^2 \frac{d\phi}{dr'} \right) r'^2 dr' = \int_0^r 4\pi G \rho(r') r'^2 dr'
\]

(15)

where

\[
M(r) = \int_0^r 4\pi G \rho(r') r'^2 dr'
\]

(16)

Giving us

\[
r^2 \frac{d\phi}{dr} = GM(r)
\]

(17)
Solution for a Hydrostatic, Isothermal Core

Now we just need to combine:

\[ r^2 \frac{d\phi}{dr} = GM(r), \quad \frac{d \ln \rho(r)}{dr} = - \frac{d}{dr} \left( \frac{\phi(r)}{c_s^2} \right) \]  
(18)

to get

\[ \frac{d \ln \rho(r)}{dr} = - \frac{GM(r)}{c_s^2 r^2} \]  
(19)

and

\[ \frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \]  
(20)

By integrating these equations, we can solve for density and mass as a function of radius for a given sound speed.
The Singular Isothermal Sphere

As a boundary condition, we must adopt a value for $\rho(0)$. If $\rho(0) \to \infty$, we get a singular isothermal sphere.

Numerical solutions:

$$\rho(r) = \frac{c_s^2}{2\pi G r^2}$$

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The Bonnor-Ebert Sphere

As a boundary condition, we must adopt a value for \( \rho(0) \). If \( \rho(0) \) is finite, we get a Bonnor-Ebert sphere.

Numerical solutions:

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The Bonnor-Ebert Sphere

Numerical solutions:

Plotted logarithmically (which we will usually do from now on)

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Numerical solutions:

Different starting $\rho_o$:

a family of solutions

$$\rho(r) = \frac{\rho_c R_c^2}{R_c^2 + r^2}$$

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The Bonnor-Ebert Sphere

Numerical solutions:

Boundary condition:
Pressure at outer edge = pressure of GMC

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Numerical solutions:

Another boundary condition:
Mass of clump is given

Must replace $\rho_c$ inner BC with one of outer BCs

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The Bonnor-Ebert Sphere

All solutions are scaled versions of each other!

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The Bonnor-Ebert Sphere

• Summary of BC problem:
  – For inside-out integration the parameters are $\rho_c$ and $r_o$.
  – However, the physical parameters are $M$ and $P_o$.

• We need to reformulate the equations:
  – Write everything dimensionless.
  – Consider the scaling symmetry of the solutions.
A dimensionless solution for the Bonnor-Ebert Sphere

Let's go back to:

$$\frac{d \ln \rho(r)}{dr} = -\frac{d}{dr} \left( \frac{\phi(r)}{\sigma_s^2} \right)$$

(21)

Let's assume $\phi(r = 0) = 0$ and define

$$\rho(r) = \rho_c e^{-\phi(r)/\sigma_s^2}$$

(22)

resulting in

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = 4\pi G \rho_c e^{-\phi(r)/\sigma_s^2}$$

(23)
if we define

\[ u \equiv \frac{\phi}{c_s^2} \]  \hspace{1cm} (24)

\[ \xi \equiv \left( \frac{4\pi G \rho c}{c_s^2} \right)^{1/2} r \] \hspace{1cm} (25)

We arrive at the Lane-Emden equation

\[ \frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{du}{d\xi} \right) = e^{-u} \] \hspace{1cm} (26)

To solve this equation, we set up the following boundary conditions

\[ u(0) = 0 \] \hspace{1cm} (27)

\[ \frac{du(\xi)}{d\xi} \bigg|_{\xi=0} = 0 \] \hspace{1cm} (28)

and then we integrate outward in \( \xi \).
A dimensionless, scale-free formulation:

A direct relation between $\rho_o/\rho_c$ and $\xi_o$

Remember:

$$\rho(r) = \rho_c \exp \left( - \frac{\Phi(r)}{c_s^2} \right)$$
We wish to find a recipe to find, for given $M$ and $P_o$, the following:

- $\rho_c$ (central density of sphere)
- $r_o$ (outer radius of sphere)
- Hence: the full solution of the Bonnor-Ebert sphere

Plan:

- Express $M$ in a dimensionless mass ‘m’
- Solve for $\rho_c/\rho_o$ (for given m)
  (since $\rho_o$ follows from $P_o = \rho_o c_s^2$ this gives us $\rho_c$)
- Solve for $\xi_o$ (for given $\rho_c/\rho_o$)
  (this gives us $r_o$)

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Now we work on defining a dimensionless mass. We do this by considering the mass of the sphere:

$$M = \int_0^{r_0} 4\pi \rho(r) r^2 dr$$ (29)

now transform into our $u$ and $\xi$ using the transformations

$$\rho(r) = \rho_e e^{-\phi(r)/c_s^2} = \rho_e e^{-u}, \quad \xi \equiv \left( \frac{4\pi G \rho_c}{c_s^2} \right)^{1/2} r$$ (30)

and get:

$$M = 4\pi \rho_c \left( \frac{c_s^2}{4\pi G \rho_c} \right)^{3/2} \int_0^{\xi_0} e^{-u} \xi^2 d\xi$$ (31)

We then plug in the Lane-Emden equation to get

$$\int_0^{\xi_0} e^{-u} \xi^2 d\xi = \xi^2 \frac{du}{d\xi}$$ (32)

which gives the mass

$$M = 4\pi \rho_c \left( \frac{c_s^2}{4\pi G \rho_c} \right)^{3/2} \xi^2 \frac{du}{d\xi}$$ (33)
The Bonnor-Ebert Sphere

We now define the dimensionless mass as

$$m \equiv \frac{P_0^{1/2} G^{3/2} M}{c_s^4}$$

(34)

or for an outer boundary $\xi = \xi_0$:

$$m = \left(4\pi \frac{\rho_c}{\rho_0}\right)^{-1/2} \left(\xi^2 \frac{du}{d\xi}\right)_{\xi_0}$$

(35)

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The Bonnor-Ebert Sphere

Dimensionless mass:

Recipe: Convert $M$ in m (for given $P_0$), find $\rho_c/\rho_o$ from figure, obtain $\rho_c$, use dimless solutions to find $r_o$, make BE sphere
Stability of Bonnor-Ebert Spheres

- Many modes of instability
- One is if \( \frac{dP_o}{d\rho_o} > 0 \)
  - Run-away collapse, or
  - Run-away growth, followed by collapse
- Dimensionless equivalent: \( \frac{dm}{d(\rho_c/\rho_o)} < 0 \)
Stability of Bonnor-Ebert Spheres

Maximum density ratio = 1 / 14.1

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Critical Pressure

In lecture 3 we found a critical radius, where $R > R_{\text{crit}}$ for a stable cloud.

$$R_{\text{crit}} = \frac{4}{15} \frac{GM_{\text{cl}}}{c_s^2}$$  \hspace{1cm} (36)

Using the equation derived from the Virial theorem

$$4\pi R_{\text{cl}}^3 P_0 = 3c_s^2 M_{\text{cl}} - \frac{3}{5} \frac{GM_{\text{cl}}^2}{R_{\text{cl}}}$$  \hspace{1cm} (37)

we derive a critical external pressure

$$P_0 = 3.15 \frac{c_s^8}{G^3 M_{\text{cl}}^2}$$  \hspace{1cm} (38)

For a Bonnor Ebert sphere:

$$R_{\text{crit}} = 0.41 \frac{GM_{\text{cl}}}{c_s^2}$$  \hspace{1cm} (39)

and

$$P_{\text{crit}} = 1.40 \frac{c_s^8}{G^3 M_{\text{cl}}^2}$$  \hspace{1cm} (40)
The Bonnor-Ebert Mass

Ways to cause BE sphere to collapse:

- Increase external pressure until $M_{BE} < M$
- Load matter onto BE sphere until $M > M_{BE}$

\[ M_{BE} = \frac{m_1 c_s^4}{P_0^{1/2} G^{3/2}} \]

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The Bonnor-Ebert Mass

Now plotting the x-axis linear (only up to $\rho_c/\rho_o = 14.1$) and divide y-axis through BE mass:

Hydrostatic clouds with large $\rho_c/\rho_o$ must be very rare...

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B68: A real Bonner-Ebert Sphere in Nature??
B68: A real Bonnor-Ebert Sphere in Nature??

$\xi_{\text{max}} = 6.9 \pm 0.2$

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$A_V = r_V^{H,K} E(H - K)$

$A_V = f N_H$

$N_H = (r_V^{H,K} f^{-1}) \cdot E(H - K)$

For optically thin emission:

$I_v = k_v \rho B_v(T_d) dl$

$I_v = m \langle k_v B_v(T_d) \rangle N_H$

$N_H = I_v [\langle m k_v B_v(T_d) \rangle]^{-1}$

$A_K = A_V \propto N_{H_2} \rightarrow n_{H_2}$

$S_v \propto T_d N_{H_2} \rightarrow n_{H_2}$

$\tau \propto N_{H_2} \rightarrow n_{H_2}$


The Dangers of Circularizing: A Faux Bonnor-Ebert Sphere

Constant Density

\( \log(\rho) \) vs. Length

\( <\log(\rho)> \) vs. Radius
Summary

Dense cores have the radii, temperatures, and masses to be thermally supported and stable.

We examine models of isothermal cores in hydrostatic equilibrium:

- singular isothermal sphere: infinite central density and outer radius

- Bonnor-Ebert sphere: finite central density and size, confined by external pressure.

Bonnor-Ebert sphere may be unstable if external pressure or mass is too high.

Actual cores have radial profiles similar to BE spheres, but BE spheres are not a unique fit to the profiles.