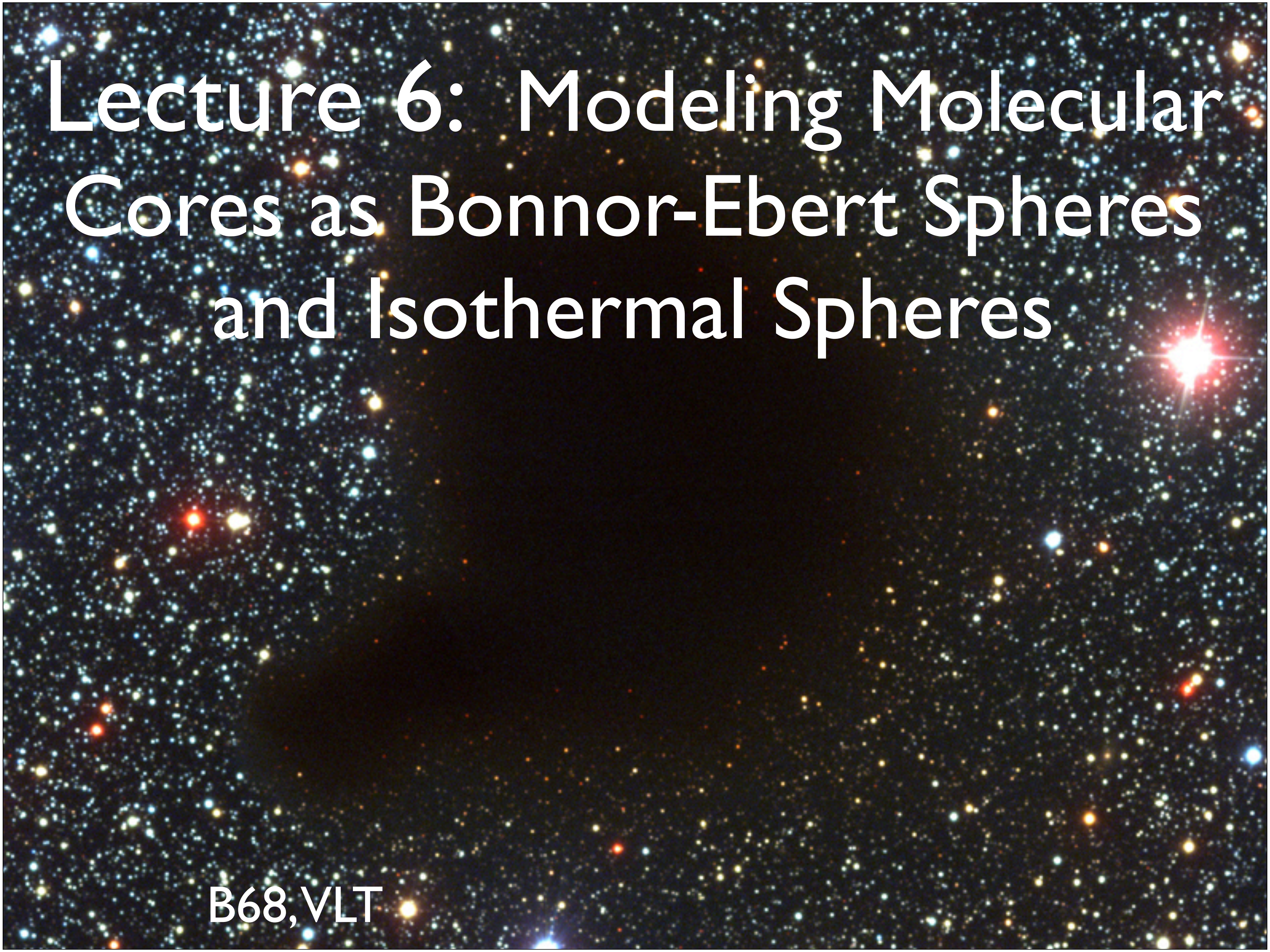
Lecture 6: Modeling Molecular Cores as Bonnor-Ebert Spheres and Isothermal Spheres



Dark Clouds in Taurus

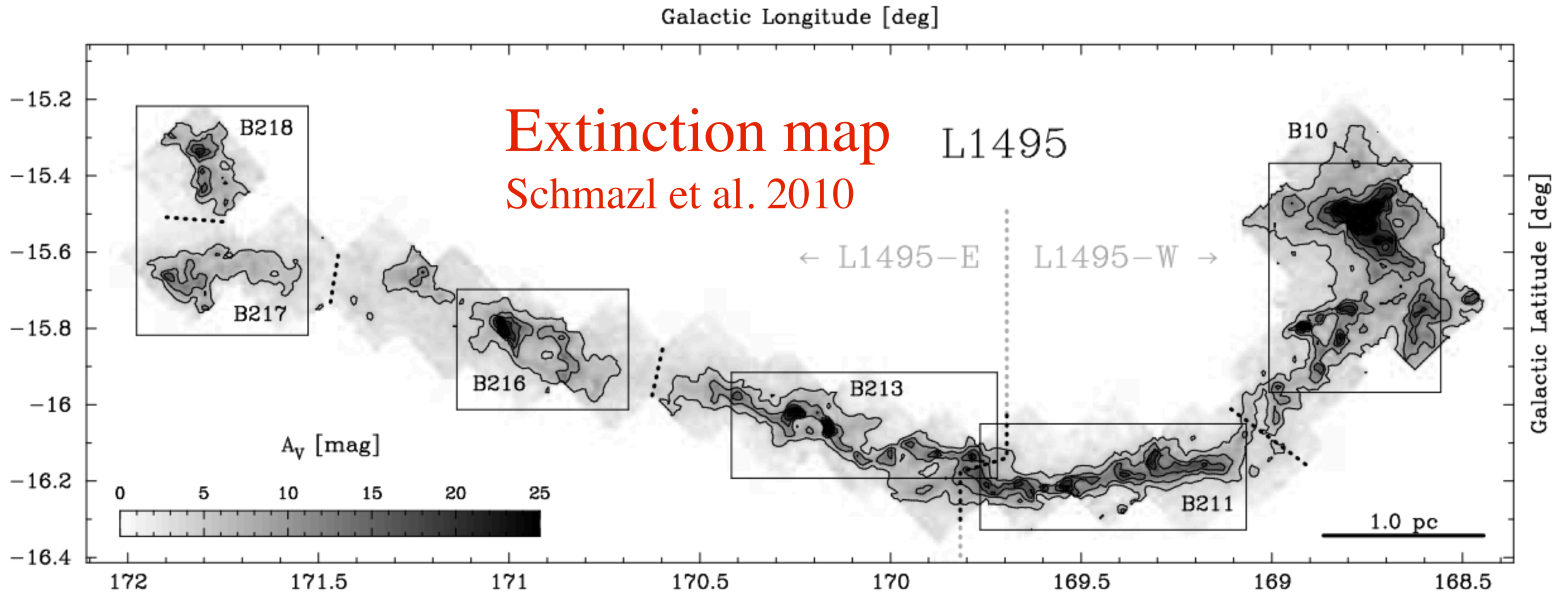


FIG. 5.— Extinction map of the Filament L 1495 with a resolution of 0.9′ derived from deep NIR observations with Omega2000. Contours are plotted in steps of $A_V = 5$ mag. We separate the filament into different subregions, which conform to Barnard's Dark Objects (Barnard et al. 1927). The boxes indicate the positions of the zoom-ins shown in Fig. 6.

Properties of Taurus Cores

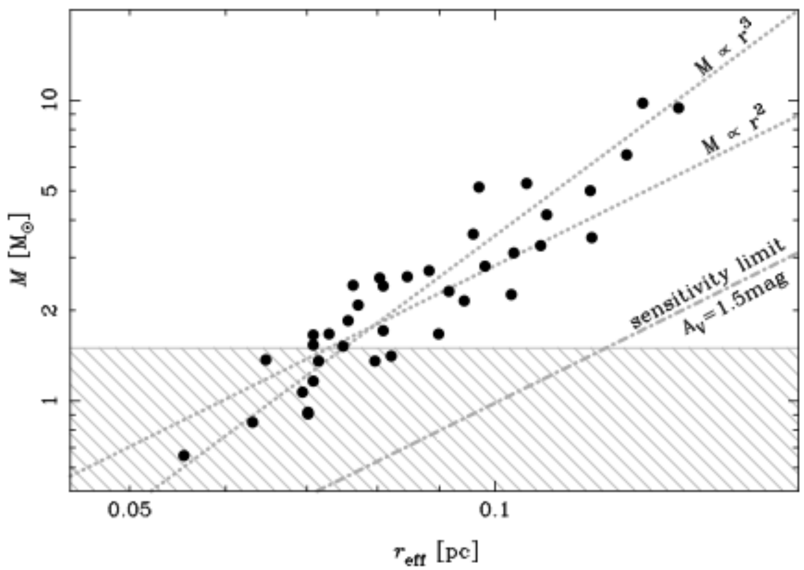


FIG. 8.— Mass-size relation for our cores. The hatched area marks the region of incompleteness. The dotted lines represent the cases of constant volume density $M \propto r^3$ and column density $M \propto r^2$.

Text

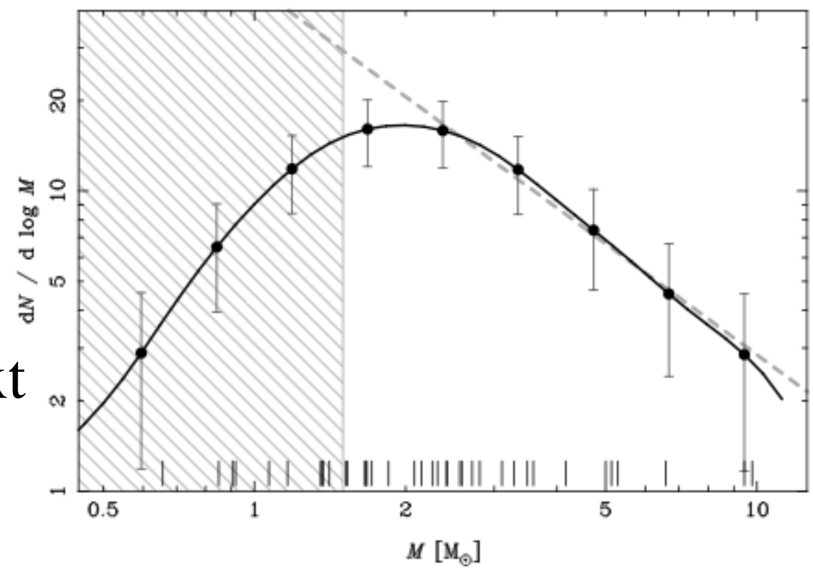


FIG. 9.— The Dense Core Mass Function (DCMF) of L 1495–E, which was obtained by smoothing the core masses (indicated as vertical dashes along the abscissa) with a Gaussian kernel with FWHM=0.3 dex. Points with error bars are plotted in separations of 50% of the FWHM. The dashed line shows the power law fit with slope $\Gamma = 1.2 \pm 0.2$ for $M > 2.0\,M_\odot$. The hatched area indicates the region of incompleteness.

Schmazl et al. 2010

Table 1: Properties of Dark Clouds, Clumps, and Cores

	Clouds[a]	Clumps[b]	Cores[c]
Mass ($M_\odot$)	10^3–10^4	50–500	0.5–5
Size (pc)	2–15	0.3–3	0.03–0.2
Mean density (cm^{-3})	50–500	10^3–10^4	10^4–10^5
Velocity extent (km s^{-1})	2–5	0.3–3	0.1–0.3
Crossing time (Myr)	2–4	≈ 1	0.5–1
Gas temperature (K)	≈ 10	10–20	8–12
Examples	Taurus, Oph, Musca	B213, L1709	L1544, L1498, B68

[a] Cloud masses and sizes from the extinction maps by Cambrésy (1999), velocities and temperatures from individual cloud CO studies

[b] Clump properties from Loren (1989) (^{13}CO data) and Williams, de Geus & Blitz (1994) (CO data)

[c] Core properties from Jijina, Myers & Adams (1999), Caselli et al. (2002a), Motte, André & Neri (1998), and individual studies using NH_3 and N_2H^+

Can Dense Cores be Supported by Thermal Pressure?

We begin by estimating the size of a thermally supported, cold (10 K), one solar mass globule of gas. The equation for hydrostatic equilibrium is

$$\frac{dP}{dr} = -\rho G \frac{M}{r^2} \tag{1}$$

or if we approximate $dP/dr = (P_c - P_0)/R$ where P_c is the central core pressure, P_0 is the outer pressure and R is the core radius, then:

$$P_c = -\rho G \frac{M}{r} \tag{2}$$

where we assume $P_0 << P_c$, and then by applying the ideal gas law ($P = c_s^2 \rho$):

$$c_s^2 \approx G\frac{M}{R} \approx \frac{kT}{\mu m_H} \tag{3}$$

Note, this is very similar to the virial equation. For a core with $M = 1M_\odot$ and $T = 10K$, we find $R = 0.15$ pc. This is very similar to the radii of molecular cores in low mass stars.

Are Dense Cores Supported by Thermal Pressure Stable?

Would such cores be stable? We can examine the critical radius for a thermally supported core with an external pressure. We found from lecture 4 that this radius would be:

$$R > R_{crit} = \frac{4}{15}\frac{GM_{cl}}{c_s^2} \tag{4}$$

For the assumed temperature and mass above, $R > 0.04pc$. Thus, the cores would be stable.

What about Jeans instabilities? The Jeans mass is given by (Lecture 3):

$$m_j = \rho_0 \lambda_J^3 = \frac{c_s^3 \pi^{3/2}}{G^{3/2}\rho_0^{1/2}} = \left(\frac{\pi k T}{\mu m_H G}\right)^{3/2} \rho_0^{-1/2} \tag{5}$$

The resulting jeans mass would be around 4 $M_{\odot}$. Thus, we have found that cores are in the regime that they could be considered to be stable condensations in hydrostatic equilibrium where the pressure is generated with thermal pressure.

Solving for Hydrostaic Equilibrium in an Isothertmal Gas: Bonnor-Ebert Spheres and Singuar Isothermal Spheres

The Spherical Core (Cow?)

We assume spherical symmetry. In this case, the term

$$\nabla P = dP/dr \tag{6}$$

where r is the radial position, and

$$\nabla^2 \phi = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) \tag{7}$$

We write the gravitational force as

$$\frac{d\phi}{dr} = -G \frac{M(r)}{r^2} \tag{8}$$

We now have three equations which describe our thermally supported gas ball. The equation for hydrostatic equilibrium becomes

We now have three equations which describe our thermally supported gas ball. The equation for hydrostatic equilibrium becomes

$$\frac{1}{\rho(r)}\frac{dP(r)}{dr} = -\frac{d\phi}{dr} \tag{9}$$

where the equation of state is the ideal gas law

$$P = \rho c_s^2 \tag{10}$$

and the equation for the gravitational potential is:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 4\pi G\rho(r) \tag{11}$$

We can combine the first two equations to find:

$$\frac{c_s^2}{\rho(r)}\frac{d\rho(r)}{dr} = -\frac{d\phi}{dr} \tag{12}$$

$$\boxed{\frac{d\ ln\ \rho(r)}{dr} = -\frac{d}{dr}\left(\frac{\phi(r)}{c_s^2}\right)} \tag{13}$$

We can integrate the equation for the gravitational potential

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 4\pi G\rho(r) \tag{14}$$

to get

$$\int_0^r \frac{1}{r'^2}\frac{d}{dr'}\left(r'^2\frac{d\phi}{dr'}\right) r'^2 dr' = \int_0^r 4\pi G\rho(r')r'^2 dr' \tag{15}$$

where

$$M(r) \equiv \int_0^r 4\pi G\rho(r')r'^2 dr' \tag{16}$$

Giving us

$$r^2\frac{d\phi}{dr} = GM(r) \tag{17}$$

Solution for a Hydrostatic, Isothermal Core

Now we just need to combine:

$$r^2 \frac{d\phi}{dr} = GM(r), \quad \frac{d\ ln\ \rho(r)}{dr} = -\frac{d}{dr}\left(\frac{\phi(r)}{c_s^2}\right) \tag{18}$$

to get

$$\frac{d\ ln\ \rho(r)}{dr} = -\frac{GM(r)}{c_s^2 r^2} \tag{19}$$

and

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \tag{20}$$

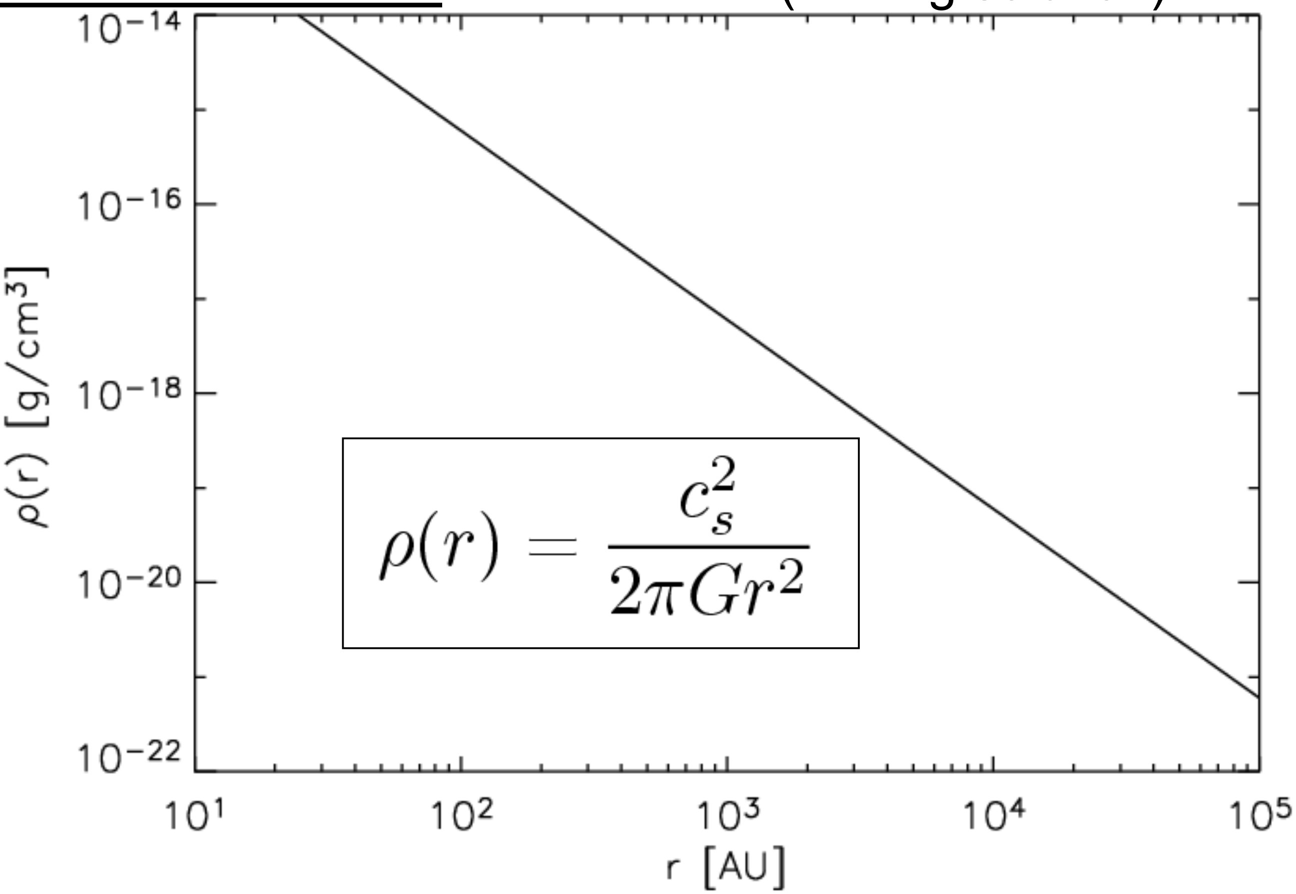
By integrating these equations, we can solve for density and mass as a function of radius for a given sound speed.

The Singular Isothermal Sphere

As a boundary condition, we must adopt a value for ρ(0).
If ρ(0) -> infinity, we get a singular isothermal sphere

Numerical solutions:

Singular isothermal sphere
(limiting solution)



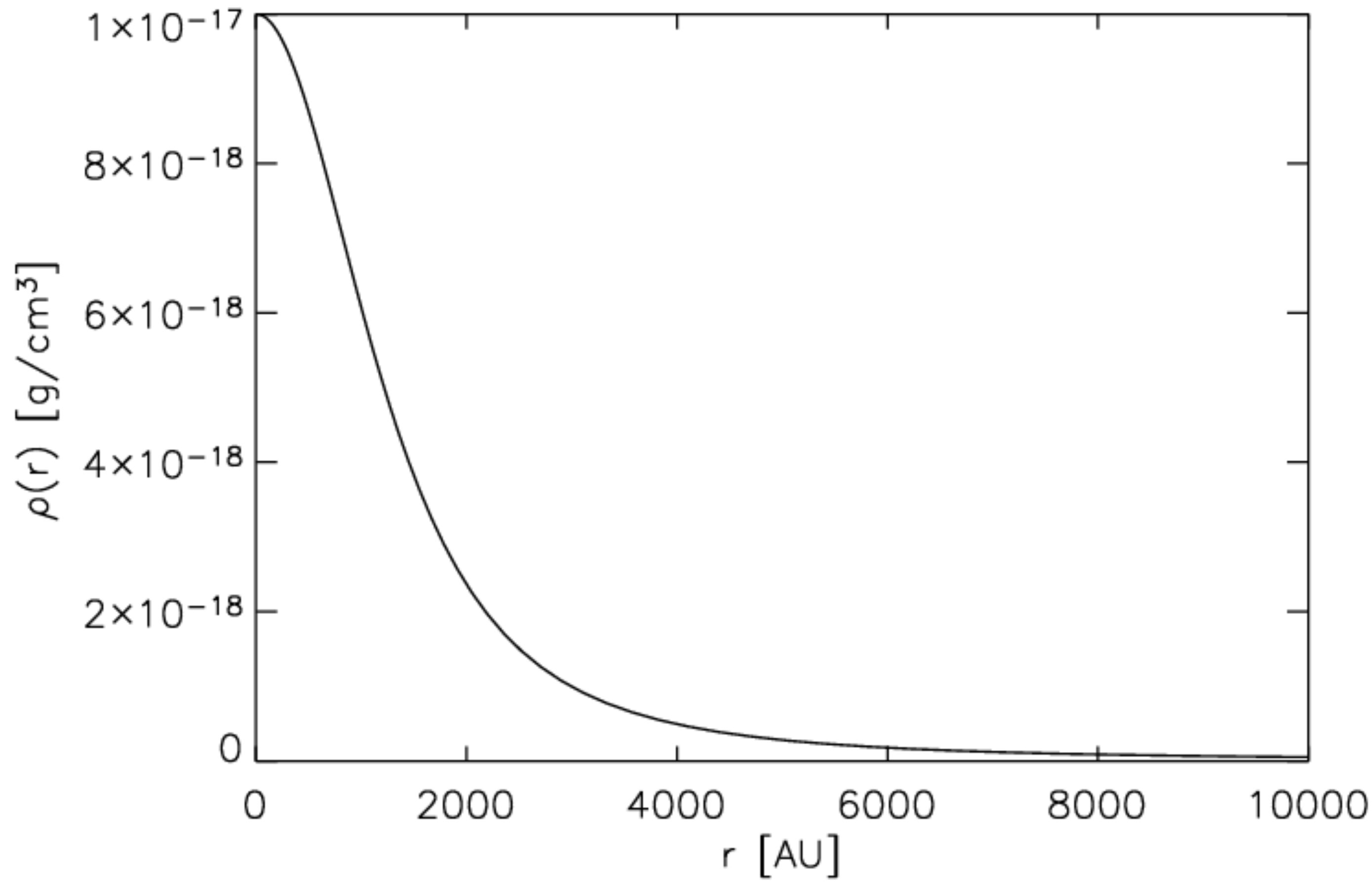
$$\rho(r) = \frac{c_s^2}{2\pi G r^2}$$

Slide pirated from K. Dullemond

The Bonnor-Ebert Sphere

As a boundary condition, we must adopt a value for ρ(0).
If ρ(0) is finite, we get a Bonnor-Ebert sphere

Numerical solutions:

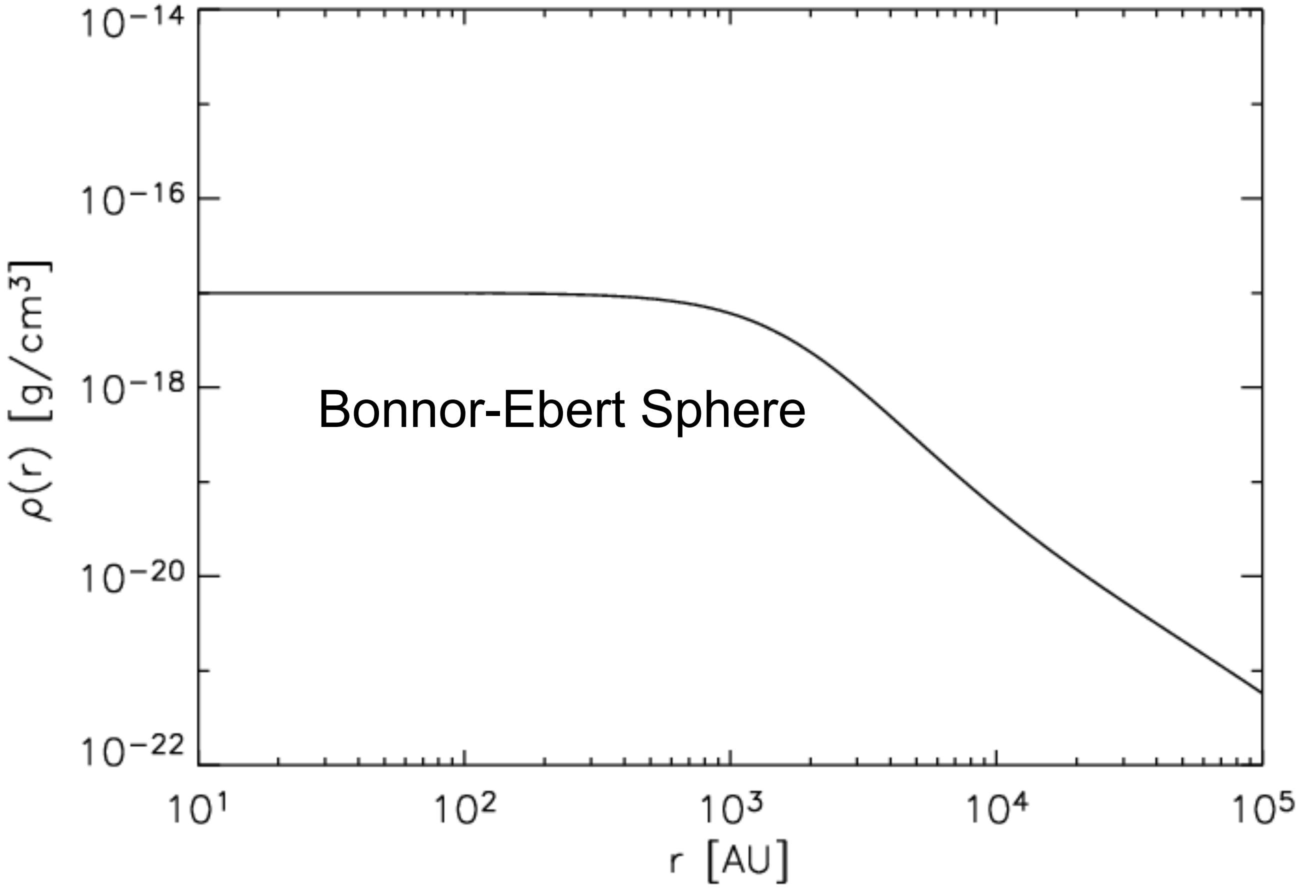


Slide pirated from K. Dullemond

The Bonnor-Ebert Sphere

Plotted logarithmically (which we will usually do from now on)

Numerical solutions:

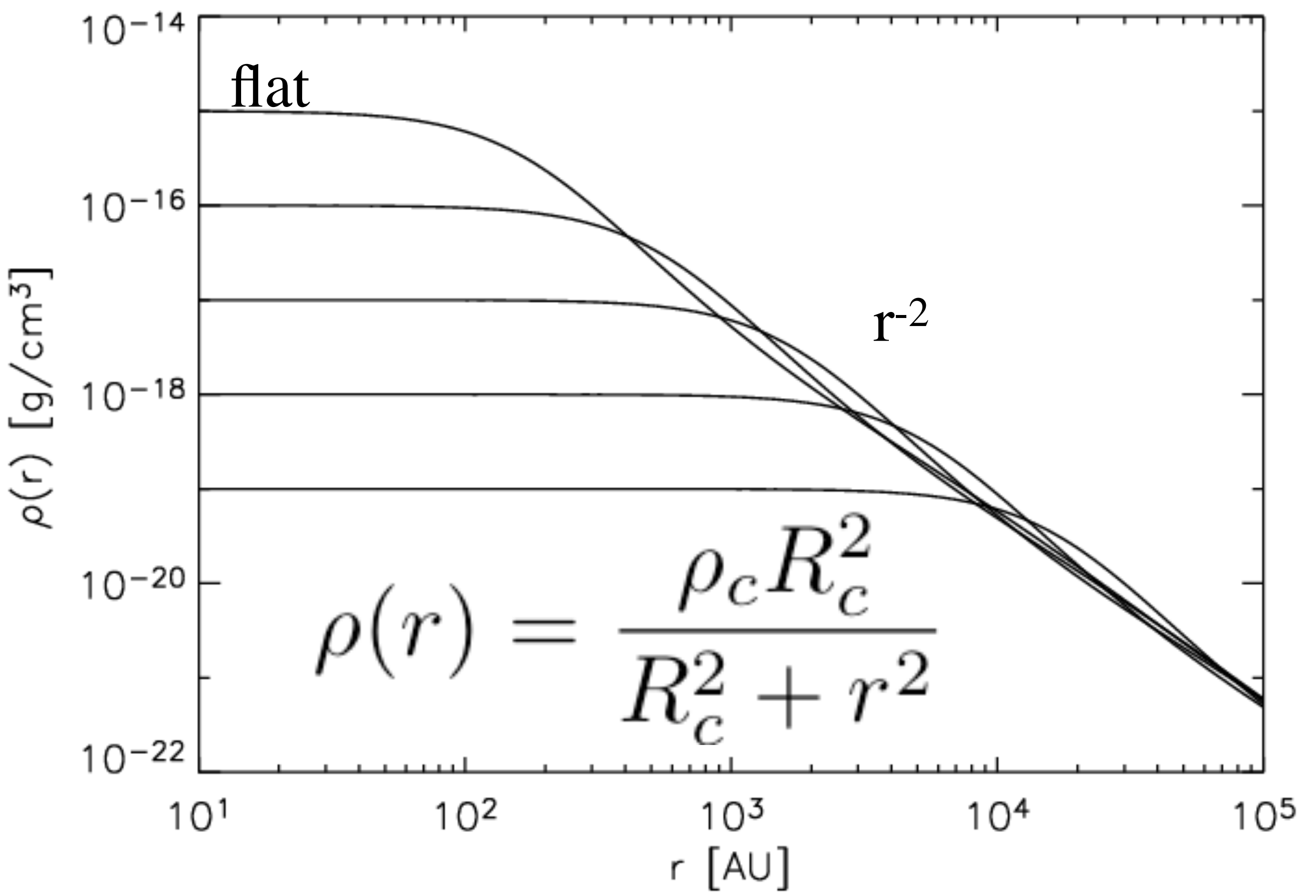


Slide pirated from K. Dullemond

The Bonnor-Ebert Sphere

<u>Numerical solutions:</u>

Different starting ρ_o :
a family of solutions

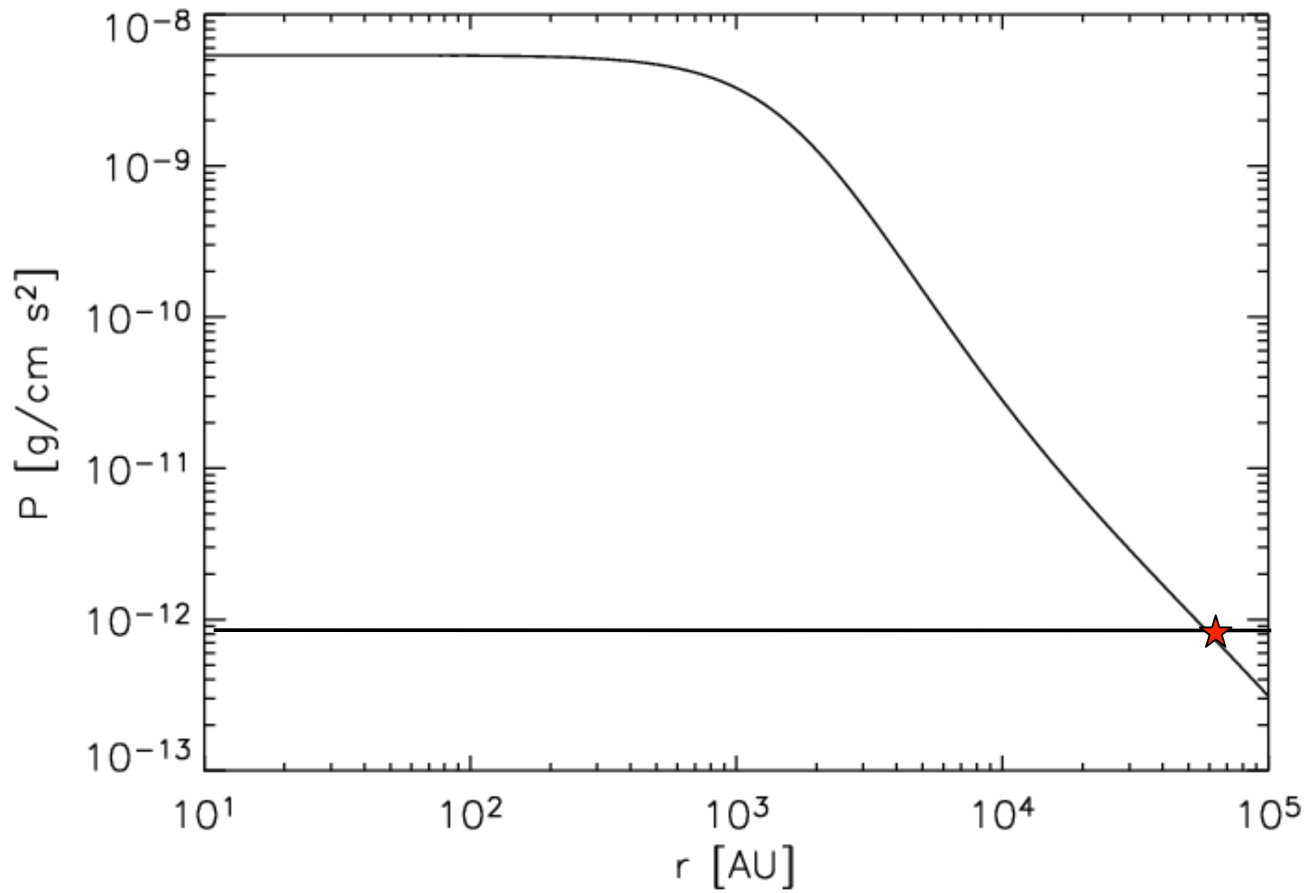


Slide pirated from K. Dullemond

The Bonnor-Ebert Sphere

<u>Numerical solutions:</u>

Boundary condition:
Pressure at outer edge
= pressure of GMC

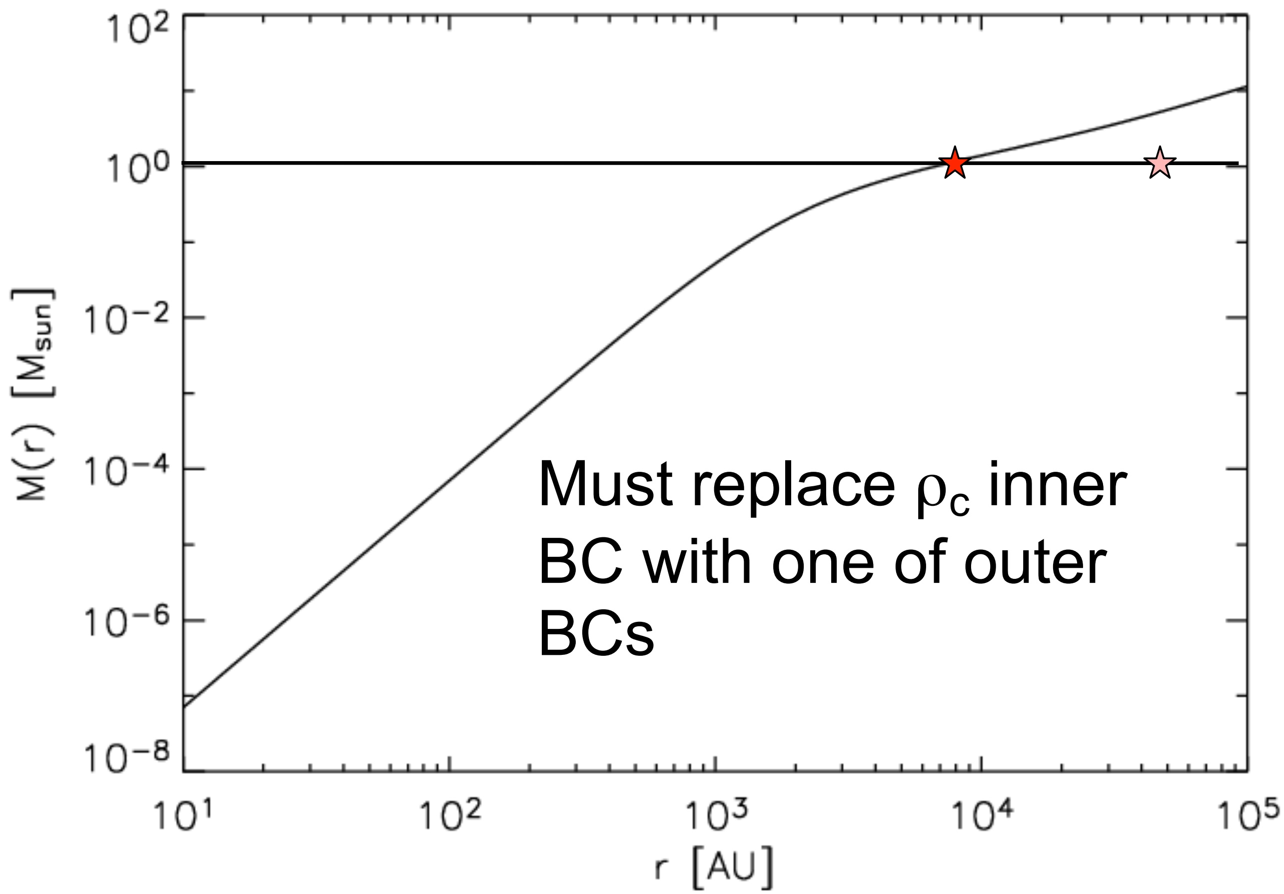


Slide pirated from K. Dullemond

The Bonnor-Ebert Sphere

Numerical solutions:

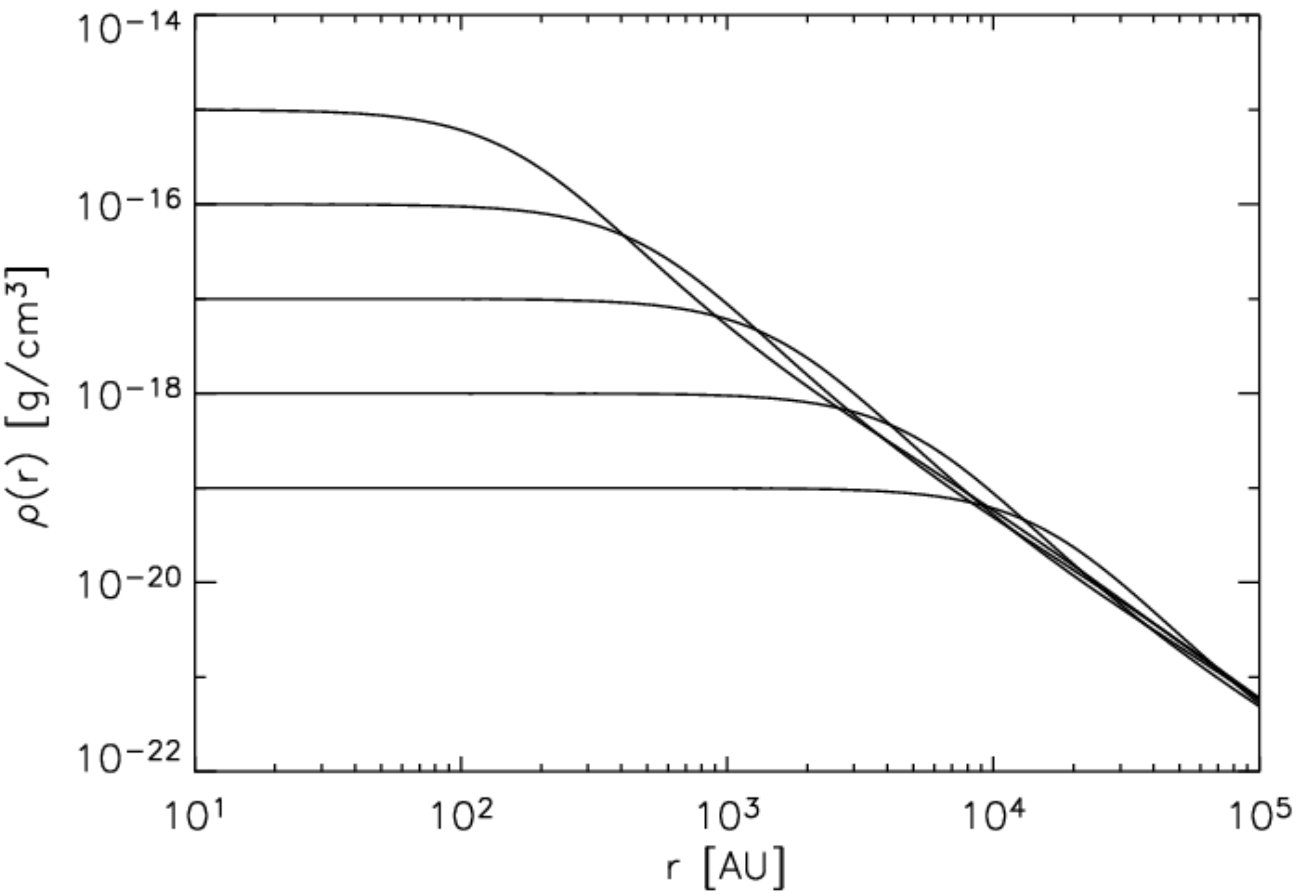
Another boundary condition:
Mass of clump is given



Slide pirated from K. Dullemond

The Bonnor-Ebert Sphere

All solutions are scaled versions of each other!



Slide pirated from K. Dullemond

The Bonnor-Ebert Sphere

- Summary of BC problem:
  - For inside-out integration the paramters are ρ_c and r_o.
  - However, the *physical* parameters are M and P_o
- We need to reformulate the equations:
  - Write everything dimensionless
  - Consider the scaling symmetry of the solutions

Slide pirated from K. Dullemond

A dimensionless solution for the Bonnor-Ebert Sphere

Let's go back to:

$$\frac{d\ ln\ \rho(r)}{dr} = -\frac{d}{dr}\left(\frac{\phi(r)}{c_s^2}\right) \tag{21}$$

Let's assume $\phi(r=0)=0$ and define

$$\rho(r) = \rho_c e^{-\phi(r)/c_s^2} \tag{22}$$

resulting in

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 4\pi G\rho_c e^{-\phi(r)/c_s^2} \tag{23}$$

if we define

$$u \equiv \frac{\phi}{c_s^2} \tag{24}$$

$$\xi \equiv \left(\frac{4\pi G\rho_c}{c_s^2}\right)^{1/2} r \tag{25}$$

We arrive at the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{du}{d\xi}\right) = e^{-u} \tag{26}$$

To solve this equation, we set up the following boundary conditions

$$u(0) = 0 \tag{27}$$

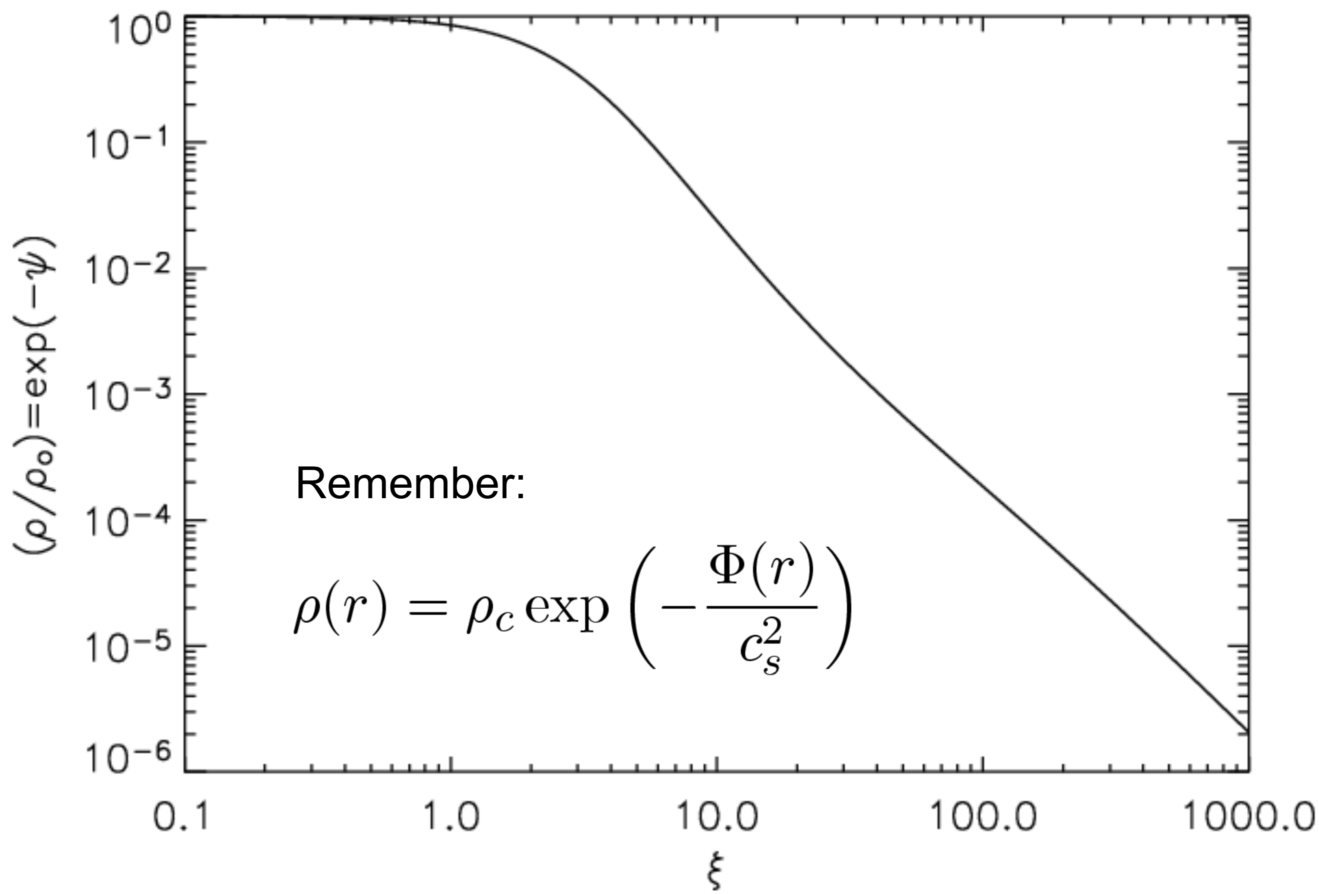
$$\frac{du(\xi)}{d\xi}|_{\xi=0} = 0 \tag{28}$$

and then we integrate outward in ξ.

The Bonnor-Ebert Sphere

<u>A dimensionless, scale-free formulation:</u>

A direct relation between ρ_o/ρ_c and ξ_o



Slide pirated from K. Dullemond

The Bonnor-Ebert Sphere

- We wish to find a recipe to find, for given M and P_o, the following:
  - ρ_c (central density of sphere)
  - r_o (outer radius of sphere)
  - Hence: the full solution of the Bonnor-Ebert sphere

- Plan:
  - Express M in a dimensionless mass ‘m’
  - Solve for ρ_c/ρ_o (for given m)

    (since ρ_o follows from $P_o = \rho_o c_s^2$ this gives us ρ_c)
  - Solve for ξ_o (for given ρ_c/ρ_o)

    (this gives us r_o)

Slide pirated from K. Dullemond

Now we work on defining a dimensionless mass. We do this by considering the mass of the sphere:

$$M = \int_0^{r_0} 4\pi\rho(r) r^2 dr \tag{29}$$

now transform into our u and ξ using the transformations

$$\rho(r) = \rho_c e^{-\phi(r)/c_s^2} = \rho_c e^{-u}, \quad \xi \equiv \left(\frac{4\pi G\rho_c}{c_s^2}\right)^{1/2} r \tag{30}$$

and get:

$$M = 4\pi\rho_c \left(\frac{c_s^2}{4\pi G\rho_c}\right)^{3/2} \int_0^{\xi_0} e^{-u}\xi^2 d\xi \tag{31}$$

We then plug in the Lane-Emden equation to get

$$\int_0^{\xi_0} e^{-u}\xi^2 d\xi = \xi^2 \frac{du}{d\xi} \tag{32}$$

which gives the mass

$$M = 4\pi\rho_c \left(\frac{c_s^2}{4\pi G\rho_c}\right)^{3/2} \xi^2 \frac{du}{d\xi} \tag{33}$$

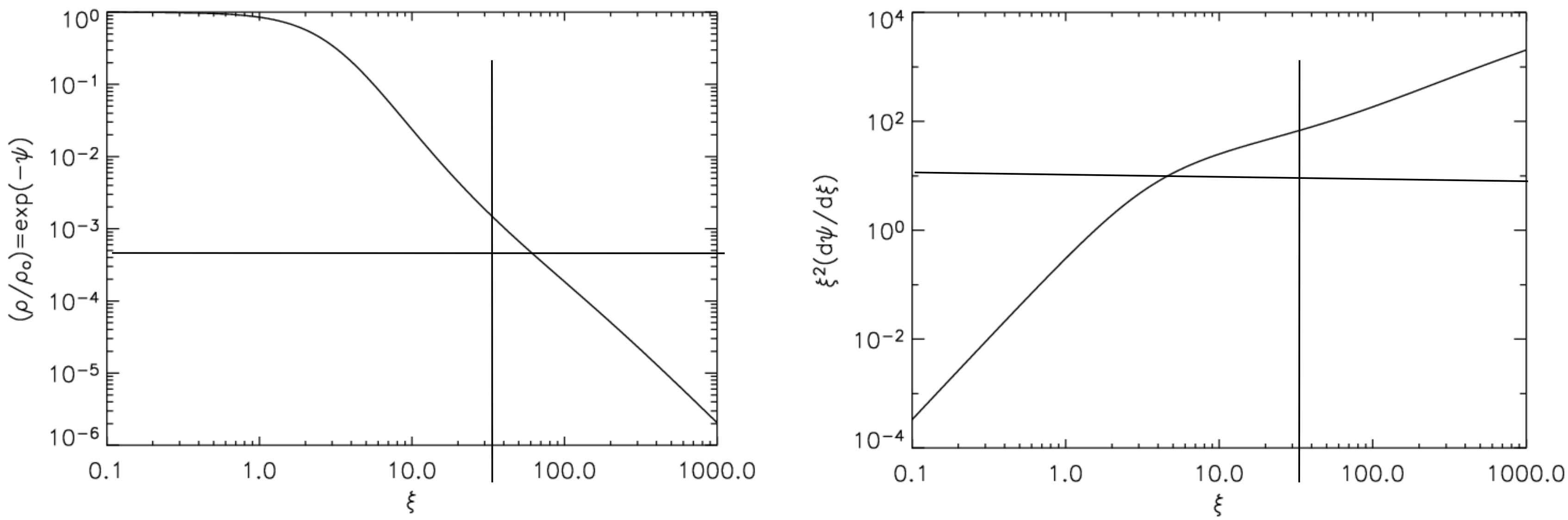
The Bonnor-Ebert Sphere

We now define the dimensionless mass as

$$m \equiv \frac{P_0^{1/2} G^{3/2} M}{c_s^4} \tag{34}$$

or for an outer boundary $\xi = \xi_0$:

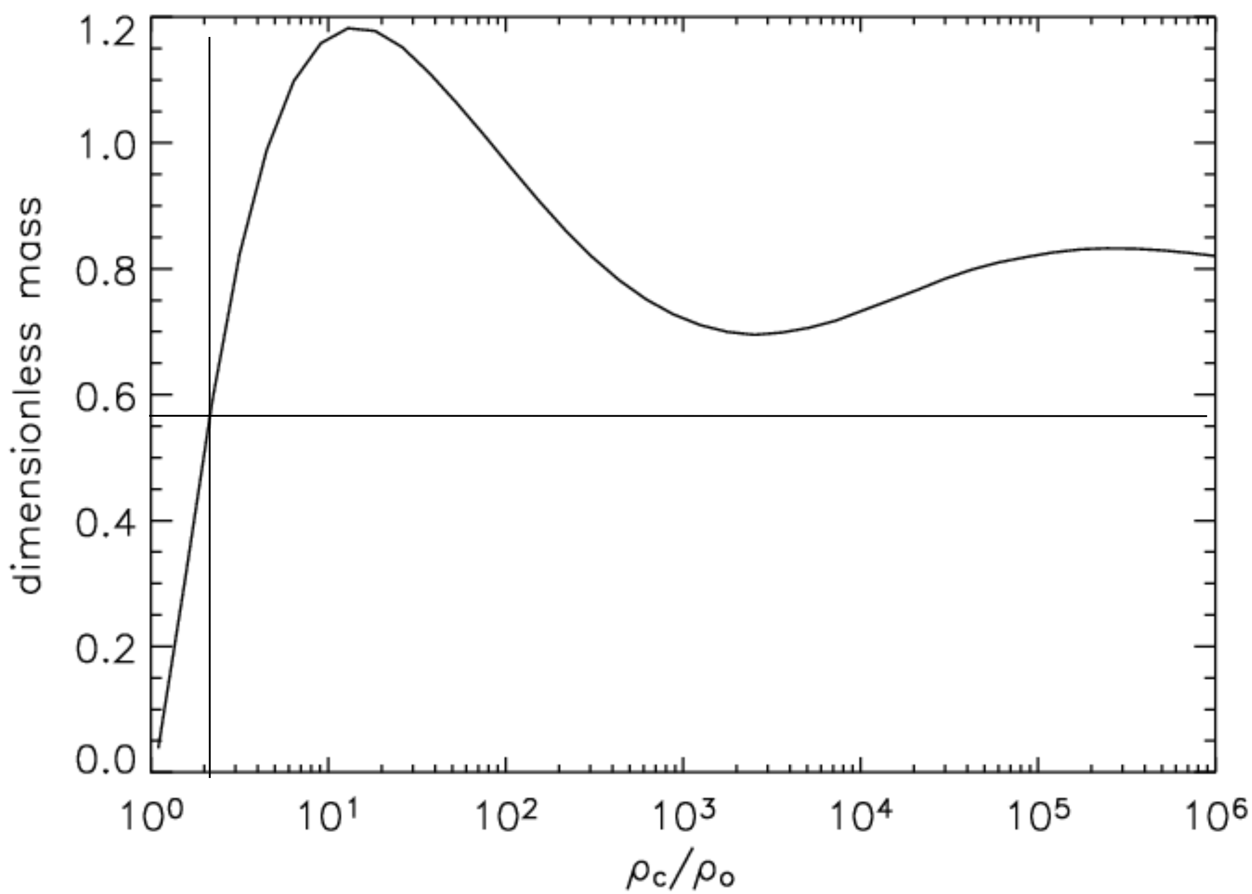
$$m = \left(4\pi \frac{\rho_c}{\rho_0}\right)^{-1/2} \left(\xi^2 \frac{du}{d\xi}\right)_{\xi_0} \tag{35}$$



Figures pirated from K. Dullemond

The Bonnor-Ebert Sphere

Dimensionless mass:

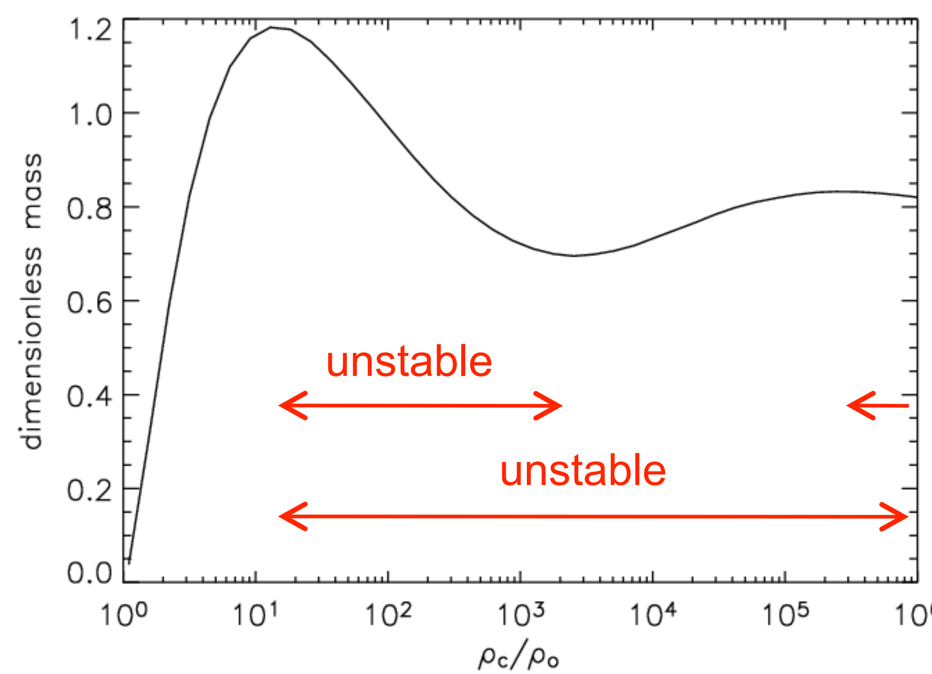


Slide pirated from K. Dullemond

<u>Recipe:</u> Convert M in m (for given P_o), find ρ_c/ρ_o from figure, obtain ρ_c, use dimless solutions to find r_o, make BE sphere

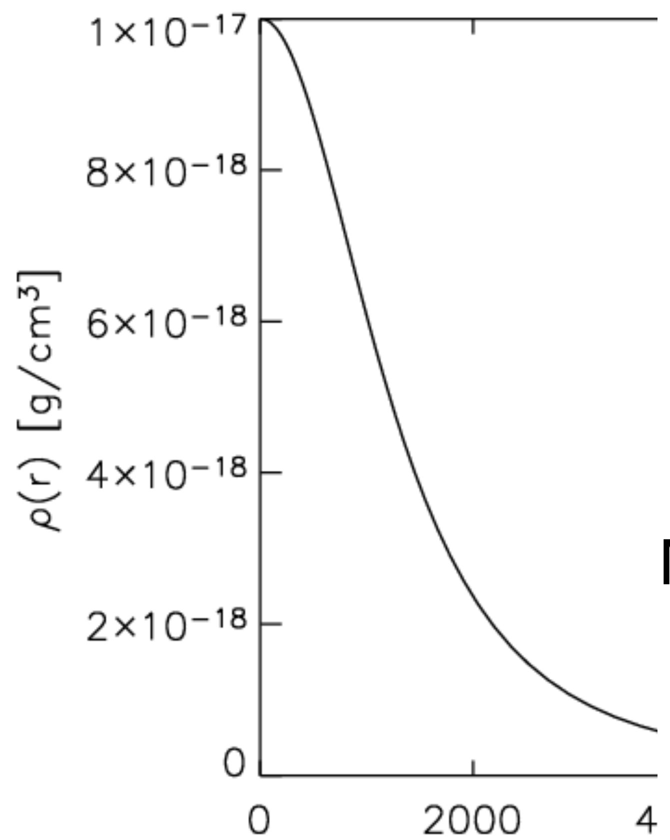
Stability of Bonnor-Ebert Spheres

- Many modes of instability
- One is if $dP_o/dr_o > 0$
  - Run-away collapse, or
  - Run-away growth, followed by collapse
- Dimensionless equivalent: $dm/d(\rho_c/\rho_o) < 0$



Slide pirated from K. Dullemond

Stability of Bonnor-Ebert Spheres



Maximum density ratio =1 / 14.1

Slide pirated from K. Dullemond

Critical Pressure

In lecture 3 we found a critical radius, where $R > R_{crit}$ for a stable cloud.

$$R_{crit} = \frac{4}{15}\frac{GM_{cl}}{c_s^2} \tag{36}$$

Using the equation derived from the Virial theorem

$$4\pi R_{cl}^3 P_0 = 3c_s^2 M_{cl} - \frac{3}{5}\frac{GM_{cl}^2}{R_{cl}} \tag{37}$$

we derive a critical external pressure

$$P_0 = 3.15\frac{c_s^8}{G^3 M_{cl}^2} \tag{38}$$

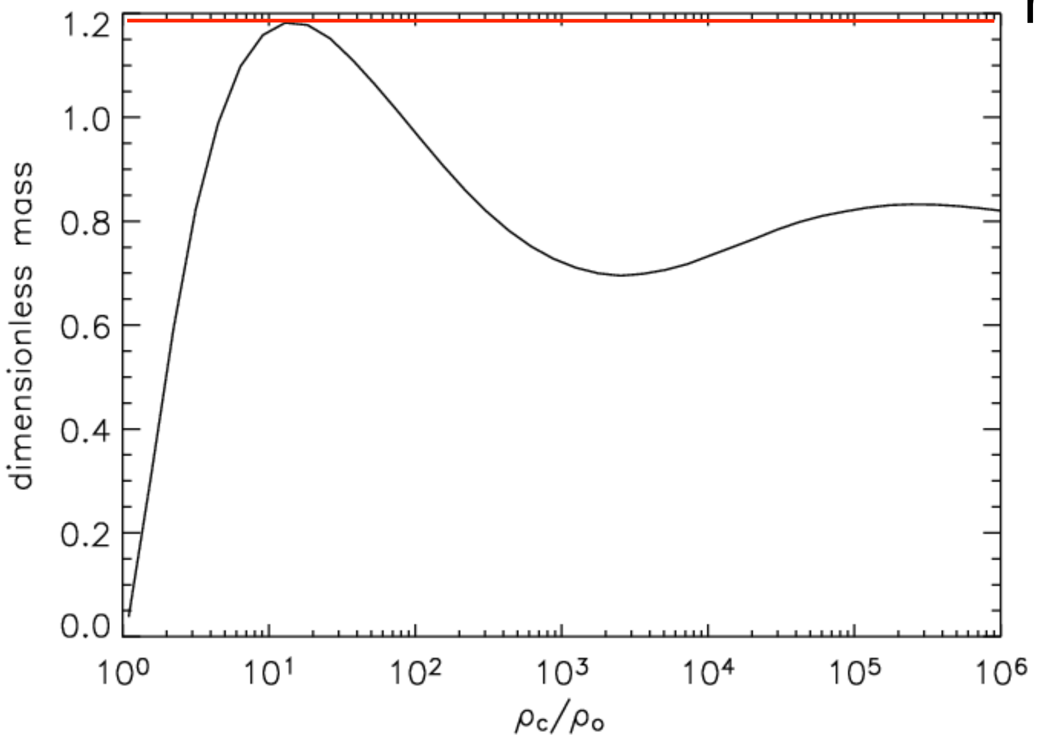
For a Bonnor Ebert sphere:

$$R_{crit} = 0.41\frac{GM_{cl}}{c_s^2} \tag{39}$$

and

$$P_{crit} = 1.40\frac{c_s^8}{G^3 M_{cl}^2} \tag{40}$$

The Bonnor-Ebert Mass



$m_1 = 1.18$

$$M_{BE} = \frac{m_1 c_s^4}{P_0^{1/2} G^{3/2}}$$

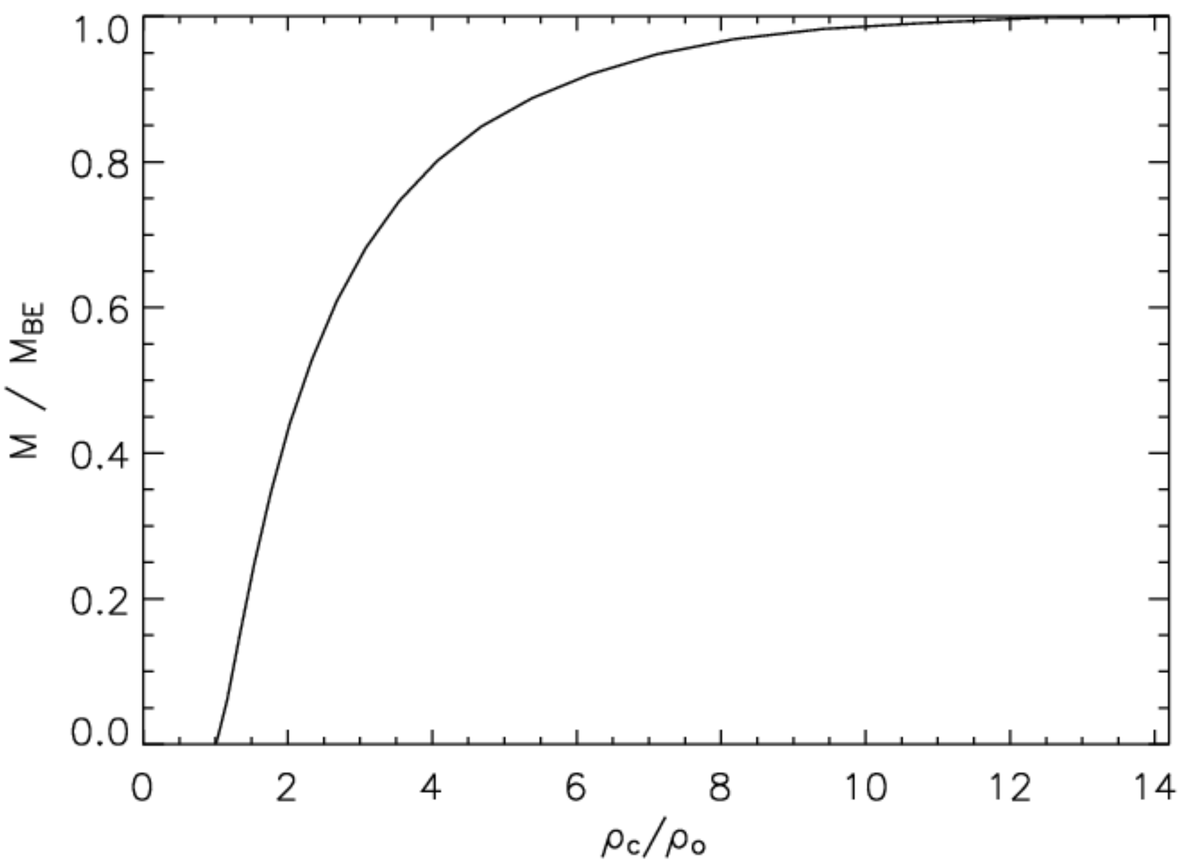
Ways to cause BE sphere to collapse:

- Increase external pressure until $M_{BE}<M$
- Load matter onto BE sphere until $M>M_{BE}$

Slide pirated from K. Dullemond

The Bonnor-Ebert Mass

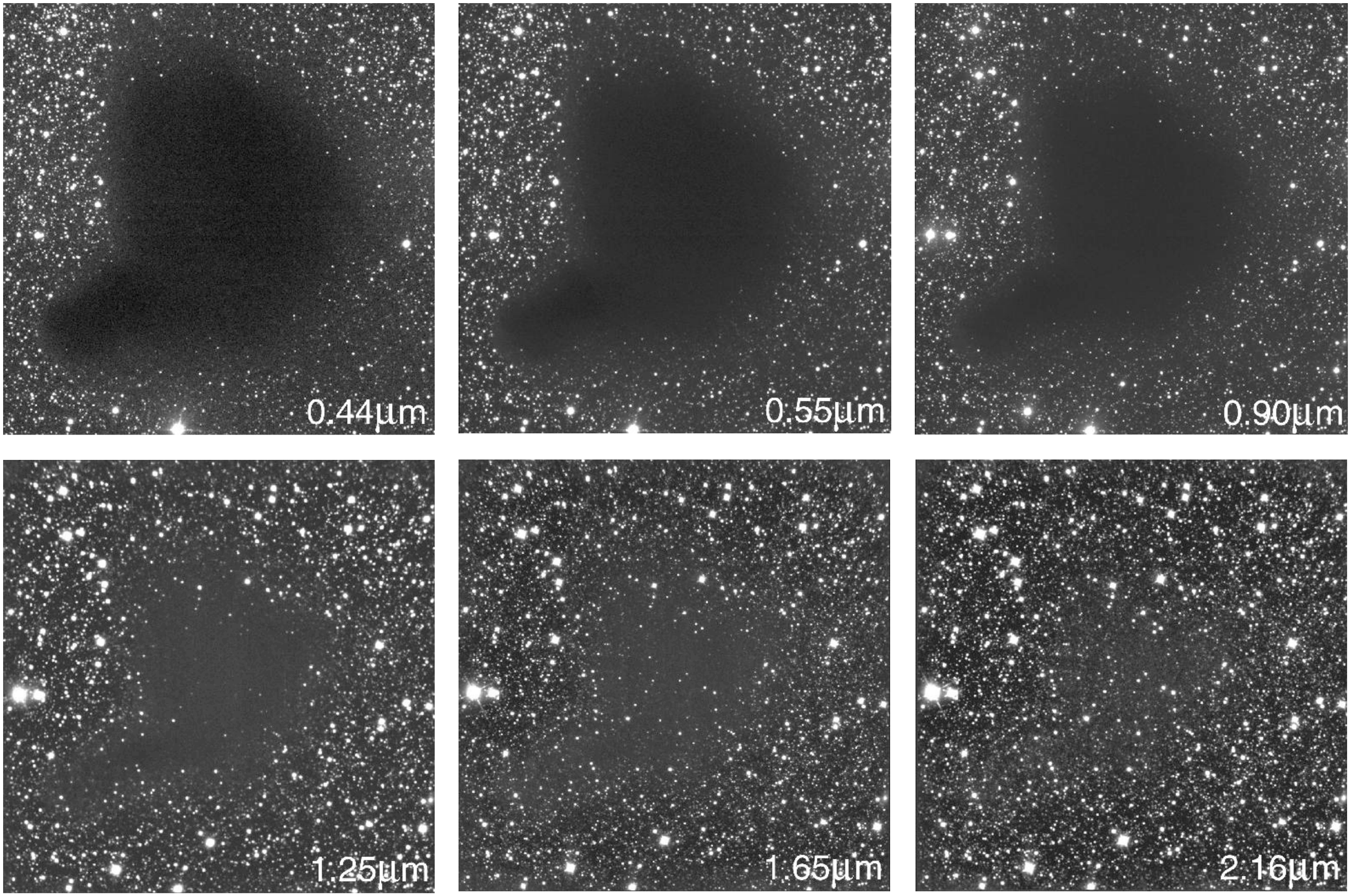
Now plotting the x-axis linear (only up to ρ_c/ρ_o =14.1) and divide y-axis through BE mass:



Hydrostatic clouds with large ρ_c/ρ_o must be very rare...

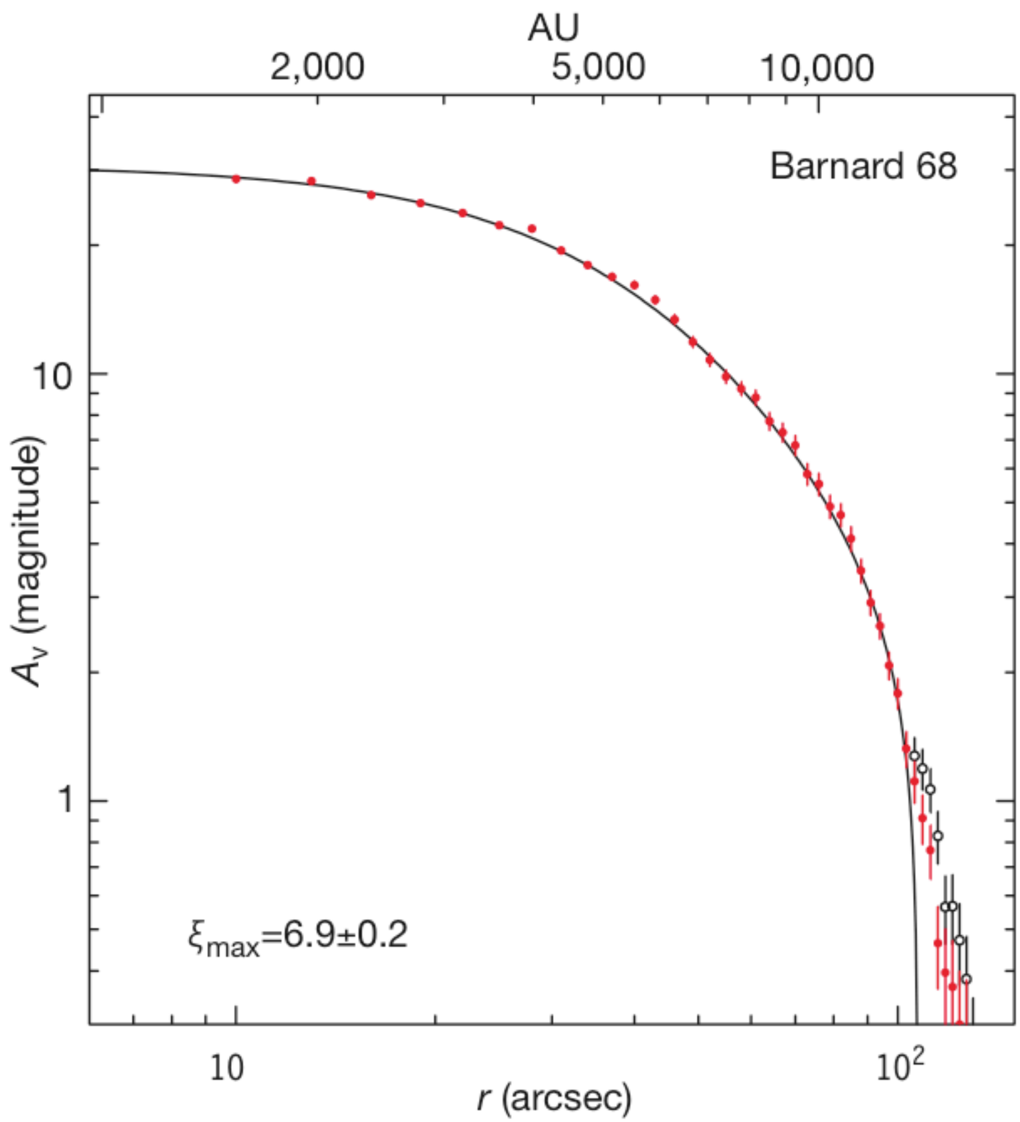
Slide pirated from K. Dullemond

B68: A real Bonner-Ebert Sphere in Nature??

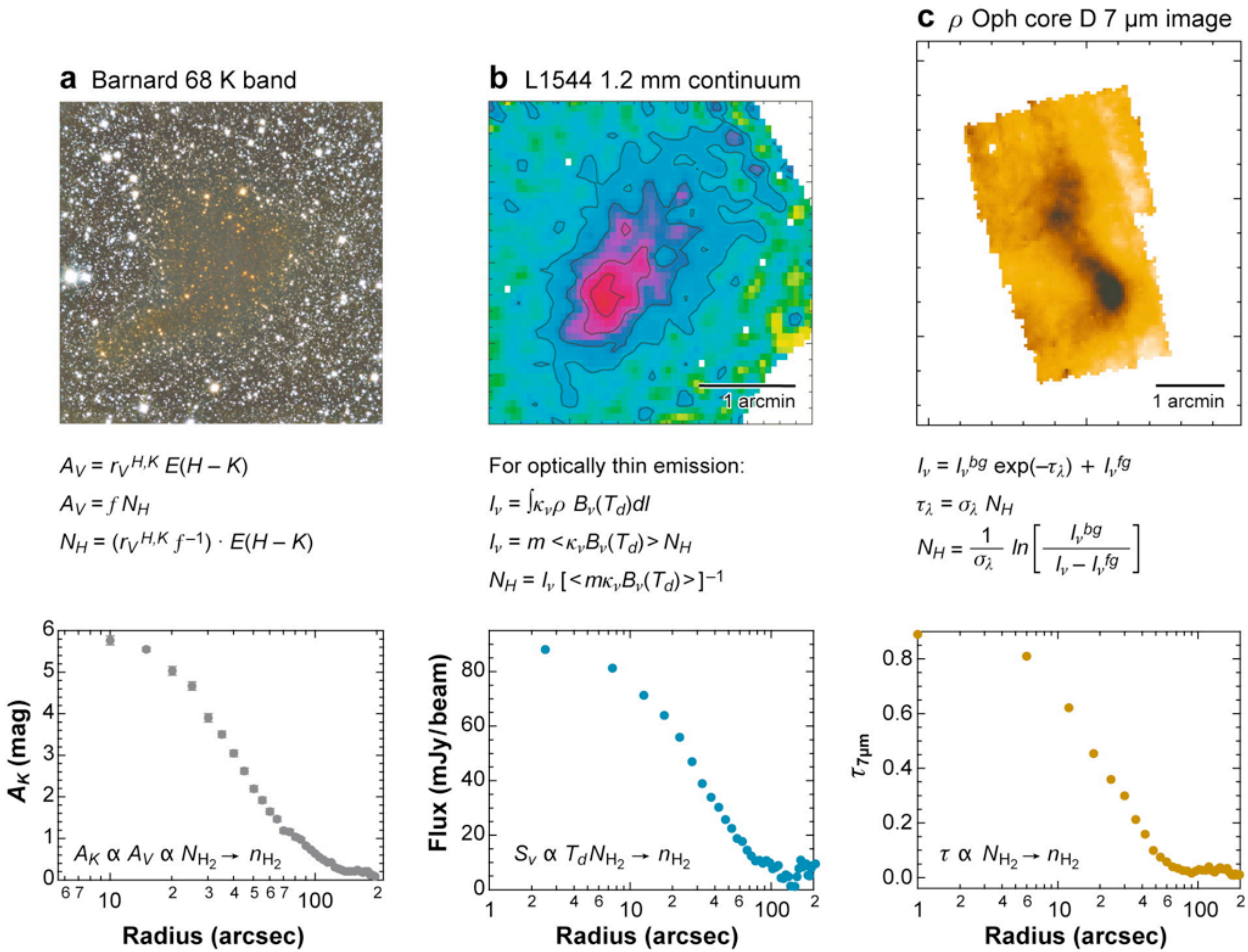


Alves, Lada, Lada 2001

B68: A real Bonnor-Ebert Sphere in Nature??



Slide pirated from K. Dullemond

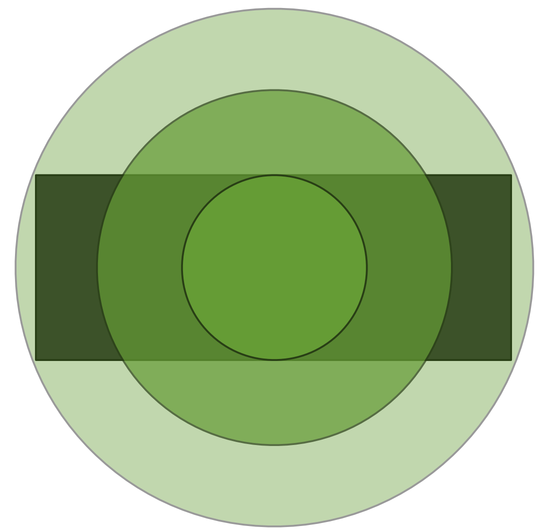


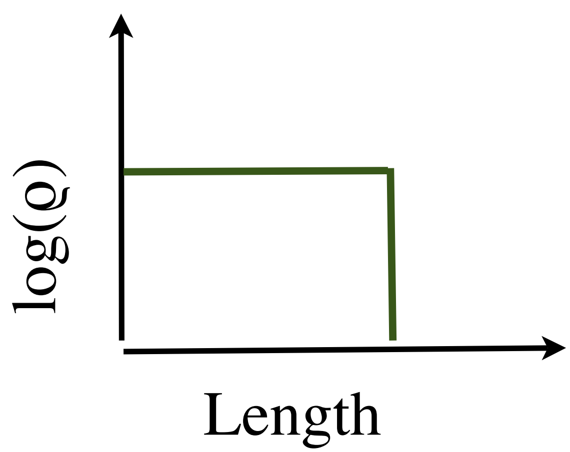
Bergin EA, Tafalla M. 2007.
Annu. Rev. Astron. Astrophys. 45:339–96

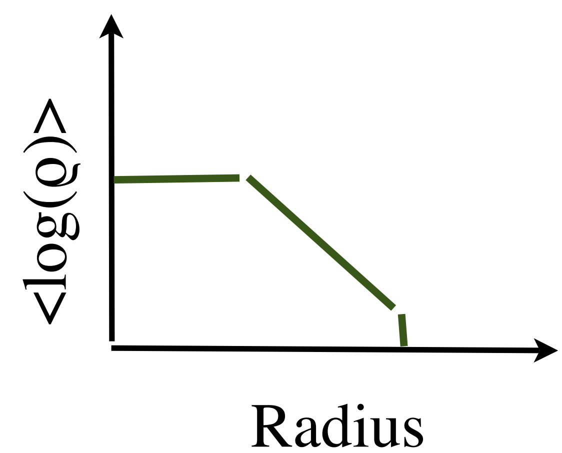
Alves et al 2001, Ward Thompson et al.. 1999, Bacman et al. 2000

The Dangers of Circularizing: A Faux Bonnor-Ebert Sphere

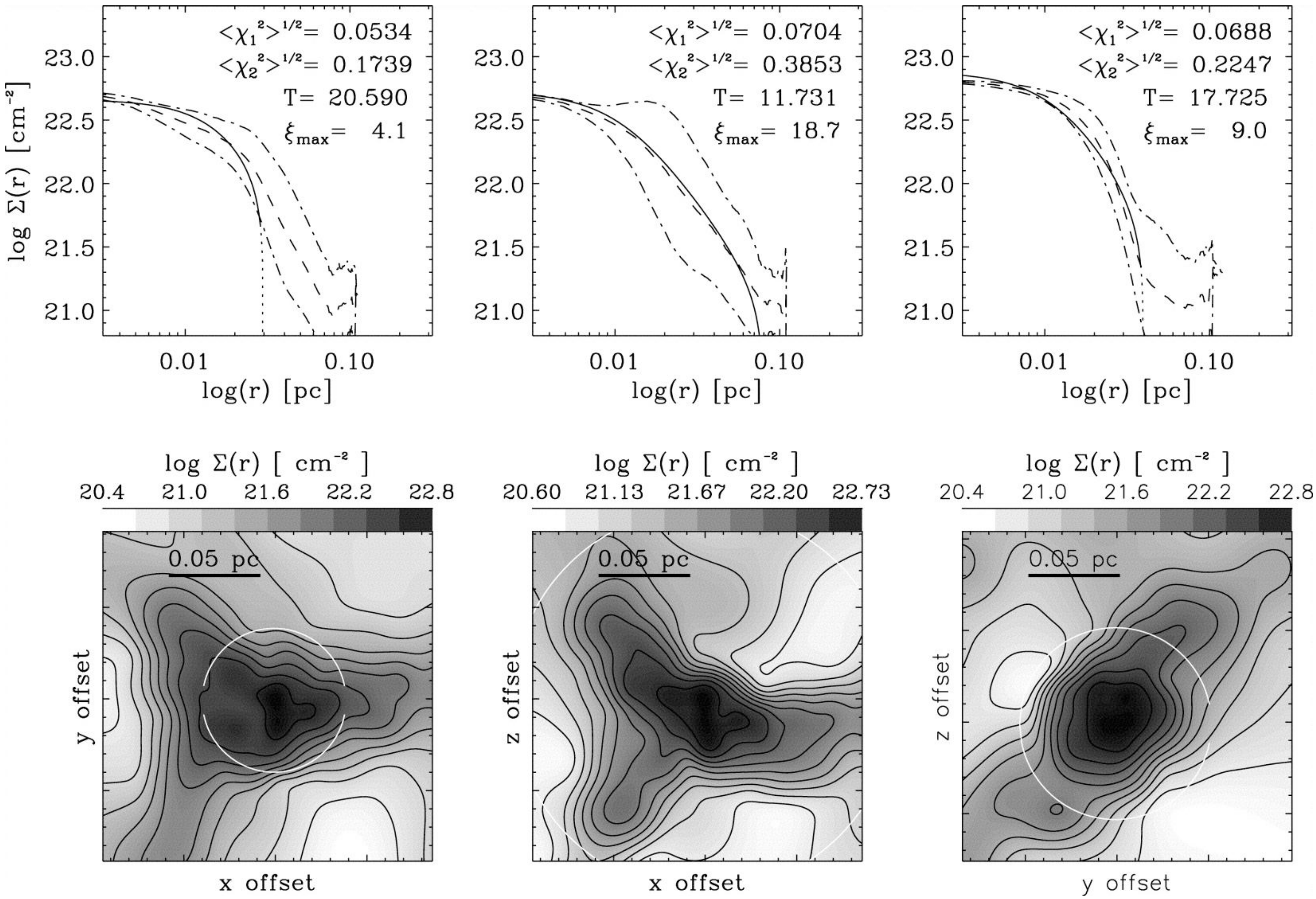
Constant Density







Dynamic Cores in Hydrostatic Digusise



Javier Ballesteros-Paredes et al. 2002

Summary

Dense cores have the radii, temperatures, and masses to be thermally supported and stable.

We examine models of isothermal cores in hydrostatic equilibrium:

> singular isothermal sphere: infinite central density and outer radius
>
> Bonnor-Ebert sphere: finite central density and size, confined by external pressure.

Bonnor-Ebert sphere may be unstable if external pressure or mass is to high.

Actual cores have radial profiles similar to BE spheres, but BE spheres are not a unique fit to the profiles.