1. *Ambipolar Diffusion (from Hartmann)*

Let's consider a gas where the number density of ions is n_i, where $n_i << n(H_2)$. The collision rate with neutrals for a given ion is

$$R_{col} = n_i < \sigma f(\mathbf{v})\mathbf{v} > \tag{1}$$

where **v** is the typical velocity difference between the particles, $f(\mathbf{v})$ is a Maxwellian distribution, σ is the cross section for collisions between ions and neutrals and $\rho/\mu m_H$ gives the density of neutrals.

Now let's assume that

1. The ions are frozen to the field.
2. The neutrals are drifting inward at a velocity v_D.
3. When a neutral collides with an ion, its drift is stopped, resulting in a change of momentum of $\mu m_H v_D$.

The change in momentum per time (i.e. the acceleration) is then given by:

$$\frac{dP}{dt} = \rho n_i < \sigma f(\mathbf{v})\mathbf{v} > v_D \tag{2}$$

To simplify the derivation, *Spitzer* originally assumed a cylindrical geometry aligned with the magnetic field. This gives a gravitational force of:

$$-\nabla\phi = 2\pi R G \rho \tag{3}$$

where R is the distance to the axis of the cylinder. We equate the force of gravity to the drag force of the ions and we get:

$$\rho \; n_i < \sigma f(\mathbf{v})\mathbf{v} > v_D = 2\pi R G \rho^2 \tag{4}$$

or

$$v_D = \frac{2\pi R G \rho}{n_i < \sigma f(\mathbf{v})\mathbf{v} >} \tag{5}$$

Finally, we can define a timescale for the diffusion of the magnetic field as $t_{amb} = R/v_D$ where:

$$t_{amb} = \frac{<\sigma f(\mathbf{v})\mathbf{v}>}{2\pi G\mu m_H}\frac{n_i}{n_H} \tag{6}$$

where $<\sigma f(\mathbf{v})\mathbf{v}> \approx 2\times 10^{-9} cm^3 s^{-1}$. This timescale depends strongly on the ratio of ions to neutrals in the cloud. We can write it approximately as:

$$t_{amb} \approx 5\times 10^{13}\frac{n_i}{n_{H_2}}\ yrs \tag{7}$$

Given estimates of $n_i/n_{H_2} = 10^{-7}(n_{H_2}/10^4 cm^{-3})$ by (McKee 1989), this gives $t_{amb} = 10^7$ years. This is longers than the lifespan of molecular clouds. Measurements also show that $\frac{N(H_2)}{B} \geq 1$, suggesting that magnetic fields do not support clouds (or cores) against collapse.

2. *Mass Infall Rate for a Thermally Supported Core*

Imaging the free fall of a core with mass M and radius R. We from the virial theorem, we showed last week for a thermally supported core:

$$c_s^2 \approx \frac{GM}{R} \tag{8}$$

Now consider the typical free fall velocity for an infalling core. Let's derive the velocity of a parcel of gas on the outside of the gas which has fallen from $R \rightarrow R/2$. The change in potential energy is:

$$\Delta U = G\frac{2mM}{R} \tag{9}$$

where m is the density of the gas. Equating this to the kinetic energy, the infall velocity is

$$v_{in} \approx 2\sqrt{\frac{GM}{R}} \approx 2c_s \tag{10}$$

Now consider the mass accretion rate. Define $t_{in} = R/c_s$. Then the mass accretion rate is given by:

$$\frac{dM}{dt} \approx \frac{M}{t_{in}} \approx \frac{Mc_s}{R} \approx \frac{c_s^3}{G} \tag{11}$$

This gives the somewhat unintuitive result that the accretion rate depends on the sound speed. Think of it this way: the higher higher the sound speed, the denser the stable core (for a given mass), and hence the higher the accretion rate.

You can derive this in another way. Take the Jeans mass:

$$m_j = \left(\frac{\pi c_s^2}{G}\right)^{3/2} \rho_0^{-1/2} \tag{12}$$

and divide it by the free fall time (Schmeja & Klessen 2004)

$$t_{ff} = \sqrt{\frac{3\pi}{32G\rho}} \tag{13}$$

to get

$$\frac{dM}{dT} \approx \frac{m_J}{t_{ff}} = \sqrt{\frac{32}{3}}\pi\frac{c_s^3}{G} \tag{14}$$

It interesting to note that in a core, where the density increases toward the center (as one would expect for a core initially in hydrostatic equilibrium) that the free fall time would decrease with decreasing R. In other words, the inner regions of the core would collapse first.

3. *A Free Falling Core*

Consider a molecular core in a state of free fall. The conservation of mass equation in a fluid can be expressed by the continuity equation:

$$\frac{\partial\rho}{\partial t} = \nabla \cdot \mathbf{v}\rho \tag{15}$$

The continuity equation for polar coordinates *and* assuming spherical symmetry is given by:

$$\frac{\partial \rho}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\ r^2 \rho v_r \tag{16}$$

Now assume a static solution where the density doesn't change quickly in time. The way to think about this is imagine are series of concentric shells centered on the protostar. The change in of mass in a given shell is equal to the flow of material into and out of the shell. If there is no change in the amount of mass in the shell ($4\pi\rho r^2 dr$), the flow in and out of the shell must be equal, i.e.

$$\frac{\partial}{\partial r}\ r^2 \rho v_r = 0. \tag{17}$$

Also assume that the mass is concentrated in the protostars. By equating the potential and kinetic energy, we obtain the free-fall velocity:

$$v_{ff} = \sqrt{\frac{2GM_\star}{R}} \tag{18}$$

then we substitute this velocity into the continuity equation to get

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\sqrt{2GM_\star}\rho r^{3/2}\right) = 0 \tag{19}$$

which implies

$$\rho \propto r^{-3/2} \tag{20}$$