

Solutions to PH6820 Midterm

1. Define the following: molecular cloud, molecular core, protostar. Include typical properties when necessary.

A *molecular cloud* is a distinct, self-gravitating cloud comprised primarily of H_2 and He. The sizes range from $10 M_\odot$ (for a dark globule) to $10^6 M_\odot$ for the most massive giant molecular clouds. Typical kinetic temperatures range from 10-50 K. The average densities are around $50\text{-}100 \text{ cm}^{-3}$. They often show filamentary and clumpy morphologies.

A *molecular core* is a self gravitating globule within a molecular cloud. The typical sizes (0.1 pc), masses (a few M_\odot), temperatures (10 K) and subsonic non-thermal velocities suggest that these are thermally supported, self gravitating entities, that will collapse to form a single star or stellar system. The presence of infrared sources in half of all cores indicates that they are often in a state of collapse.

A *protostar* is a star undergoing rapid mass accretion. They are found in the middle of collapsing envelopes that absorb their luminosity and reprocess it into the infrared.

2. Imagine a protostar with a constant infall rate given by the solution for an infalling Singular Isothermal Sphere.

a. Assuming the infall rate is equal to the accretion rate, how does the mass of the central star change with time? Write the answer in terms of the sound speed.

Solution: The infall rate of a collapsing, thermally supported, core is given by (approximately!):

$$\dot{M} = \frac{c_s^3}{G} \quad (1)$$

Thus, the total mass accumulated is simply the infall rate time the time over which infall has occurred:

$$M = \frac{c_s^3}{G} t \quad (2)$$

where t is the age of the protostars. I'm assuming that the initial mass is negligible, but there probably had to be some initial mass.

b. Assuming the central star has a constant density, how does accretion luminosity vary with time? Write the answer in terms of the sound speed, and the density of the star.

Solution: The accretion luminosity is given by:

$$L_{acc} = \frac{G\dot{M}M}{R} \quad (3)$$

What is the radius? If we assume a constant density, $R = (3M/4\pi\rho)^{1/3}$. Thus, the luminosity is

$$L_{acc} = G \left(\frac{c_s^3}{G} \right) \left(\frac{c_s^3 t}{G} \right) \left(\frac{4\pi G \rho}{3c_s^3 t} \right)^{1/3} = (4\pi/3)^{1/3} G^{-2/3} c_s^5 \rho^{1/3} t^{2/3} \quad (4)$$

c. Also assume that the intrinsic luminosity of the central protostar is given by the relationship $L = L_\odot (M/M_\odot)^{3.5}$? At what mass does the intrinsic luminosity of the source equal the accretion luminosity? Adopt a density of 3.5 gm cm^{-3} . Also assume the gas temperature is 20 K and $\mu = 2.7$.

Solution: The sound speed is given by $c_s = \sqrt{kT/\mu m_H} = 0.25 \text{ km s}^{-1}$. With a density of 3.5 gm cm^{-3} , this gives us:

$$L_{acc} = 77 \left(\frac{t}{10^5 \text{ yrs}} \right)^{2/3} L_\odot \quad (5)$$

while the intrinsic luminosity is:

$$L_{int} = \left(\frac{c_s^3 t}{GM_\odot} \right)^{3.5} L_\odot = 0.02 \left(\frac{t}{10^5 \text{ yrs}} \right)^{3.5} L_\odot \quad (6)$$

then $L_{int} = L_{acc}$ means

$$0.02 \left(\frac{t}{10^5 \text{ yrs}} \right)^{3.5} = 77 \left(\frac{t}{10^5 \text{ yrs}} \right)^{2/3} \quad (7)$$

which has the solution of $t = 1.8 \text{ Myr}$. At this point, the Mass is then given by equation 2.

$$\frac{c_s^3}{G} t = \left(\frac{(2.5 \times 10^4 \text{ cm s}^{-1})^3}{6.8 \times 10^{-8} \text{ cm}^3 \text{ gm}^{-1} \text{ s}^{-2}} \right) 1.8 \text{ Myr} \cdot 10^6 \text{ yr} \cdot \pi \times 10^7 \text{ s} = 2 M_\odot \quad (8)$$

3. In the very early universe, the primary coolant of molecular clouds is molecular hydrogen (there were few if any metals). Thus, molecular clouds would be much warmer since H_2 does not have any low-lying energy levels. Lets adopt a cloud temperature of 100 K.

a. Assuming that the gas is molecular hydrogen and helium with the current abundance of He ($\mu = 2.7$), what is the typical sound speed?

Solution: the sound speed is given by:

$$c_s = \sqrt{\frac{kT}{\mu m_H}} = \sqrt{\frac{1.38 \times 10^{-16} \text{ erg } K^{-1} 100 \text{ K}}{2.7 \cdot 1.66 \times 10^{-24} \text{ gm}}} \quad (9)$$

b. What would be the Jeans mass for a $10,000 \text{ cm}^{-3}$ molecular cloud?

Solution:

$$M_J = \pi^{3/2} c_s^3 (G^3 \rho)^{-1/2} = 200 M_\odot \quad (10)$$

c. How would it compare to that in a typical molecular cloud today?

Solution: It would be significantly greater, at least by a factor of 10 (for a cloud with gas temperatures of 20 K)

d. How would infall rates differ between protostars now and then?

Solution: The infall rates would be significantly higher. Since the sound speed is a factor of 2.2 greater, the infall rate (c_s^3/G) would be 11 times greater.

e. How might stars and the initial mass function be different?

Solution: The masses of the first star are expected to be systematically higher due to the larger Jeans mass and the higher infall rates.

4. Below are millimeter-wave spectra toward protostars. They are observed in three lines: top is a spectrum of the molecule CS (partially optically thick), the middle is a spectrum of the molecule H_2CO (partially optically thick) and the bottom is a line of the molecular ion H_2H^+ (optically thin). Which of the following cores show infall, which show outflow, and which show neither?

Solution: Red-shifted self-absorption implies infall, blueshifted self-absorption implies outflows. These sources have redshifted self-absorption (i.e. exhibit infall): L1172, L1251a, L1251b, L1152. The following show blue-shifted self-absorption (i.e. exhibit outflow): L43, L146 and L1262. The following exhibit neither: IRS32.

5. Define the following: Pre-main sequence star, T Tauri star, Herbig Ae/Be star.

Pre-main sequence stars: the pre-main sequence phase occurs after the protostellar phase and before the main sequence. During the pre-main sequence, the star has accreted most of its mass, but does not yet have central temperatures needed for Hydrogen fusion to occur.

T Tauri star: a low mass pre-main sequence star. Classical T Tauri stars have disks, are accreting small amounts of gas, and often show strong line emission from accretion and winds. Weak line T Tauri stars do not have disks and do not show strong line emission.

Herbig Ae/Be star: an intermediate mass pre-main sequence star. These stars show emission line (by definition) and have disks. The original definition by George Herbig was an A or B spectral type, strong line emission, and proximity to nebulosity (to show that they are near their natal molecular gas).

6. a. Calculate the energy released in an accretion disk as gas moves from a radius of $r + dr$ to r . Assume that the gas is in Keplerian rotation.

Solution: In Keplerian rotation, the velocity of a rotating parcel of gas is $v_\theta = \sqrt{GM_\star/R}$. Thus, an annulus of gas with mass M in Keplerian rotation would have the total energy

$$E = \frac{1}{2}dMv_\theta^2 - G\frac{M_{star}}{r} = -\frac{1}{2}G\frac{M_\star dM}{r} \quad (11)$$

and the change in energy is then:

$$dE = \frac{1}{2}\frac{GM_\star dM}{r^2}dr \quad (12)$$

b. Now, converting dr/dt to \dot{M} , show that the luminosity of a given annulus in an accretion disk is given by $L = GM\dot{M}/2R^2$. Give the details of your derivation. At what radius does most of the luminosity come from?

Solution: Let's consider a disk with a surface density Σ . Then the mass in a given ring is $dM = \Sigma 2\pi r dr$. We can then write the mass flow at a given radius as

$$dM = \Sigma 2\pi r dr \text{ and } \dot{M} = \Sigma 2\pi r \frac{dr}{dt} = \Sigma 2\pi r v_r \quad (13)$$

where v_r is the velocity of the gas in the radial direction. The rate of energy loss is then

$$\dot{E} = \frac{1}{2}\frac{GM_\star \Sigma \pi r dr}{r^2}v_r = \frac{1}{2}\frac{GM_\star \Sigma \pi r v_r}{r^2}dr = \frac{1}{2}\frac{GM_\star \dot{M}}{r^2} \quad (14)$$

Note that v_r would be negative in this case.

c. Derive the rate at which the same annulus loses angular momentum in terms of G , M , \dot{M} and R .

Solution: the angular momentum of the ring is given by:

$$L_{ang} = dM v_{\theta} r = \Sigma r^2 \sqrt{\frac{GM_{\star}}{r}} dr = \Sigma \sqrt{GM_{\star} r^3} dr \quad (15)$$

we can then write the change in angular momentum as

$$\dot{L}_{ang} = \Sigma \left(\frac{3GM_{\star} r}{2} \right)^{1/2} dr v_r = \Sigma r \left(\frac{3GM_{\star}}{2r} \right)^{1/2} dr v_r = \dot{M} \left(\frac{3GM_{\star}}{2r} \right)^{1/2} dr \quad (16)$$

7. Consider a 1 solar luminosity star with a narrow belt of planetesimals at 1 A.U. from the star.

a. Calculate the temperature of the planetesimals. Assume that the planetesimals are spherical and grey: they reflect 10% of the incoming radiation and absorb the rest at all wavelengths.

Solution: The temperature is given by the balance of absorption and emission. For an absorption of a_{λ} of 0.9 at visible light and $\epsilon_{\lambda} = 1$ at infrared wavelengths, and assuming a grain radius of r and a luminosity and distance to the star of L and R

$$a_{\lambda} \pi r^2 \frac{L}{4\pi R^2} = \epsilon_{\lambda} 4\pi r^2 \sigma T^4 \quad (17)$$

resulting in:

$$T = \left(\frac{a_{\lambda} L}{\epsilon_{\lambda} 16\pi \sigma R^2} \right)^{1/4} = 273 \text{ K} \quad (18)$$

b. Next calculate the wavelength of peak flux density, F_{λ} , for the star and for the planetesimals. Adopt a temperature of 6000 K for the stellar photosphere.

Solution: use Wien's law that $\lambda T = 2900 \mu\text{m K}$ giving $\lambda = 10 \mu\text{m}$.

c. If the planetesimals were each 1 km in diameter, write an expression for the ratio of the luminosity density, L_{λ} (the luminosity per wavelength, this is just $F_{\lambda} 4\pi D^2$ where D is the distance of the star to the Earth) of the belt relative to the star at the wavelength of peak

flux density for the belt. Write this in terms of the total combined mass of the asteroids, the density of the asteroids (5 gm cm^{-3}), the diameter of the asteroids, the luminosity, temperature and radius of the star, and the distance of the belt to the star.

Solution: For a blackbody star, L_λ is given by:

$$L_\lambda(\text{star}) = 4\pi R_\star^2 \pi \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT_\star} - 1} = 4\pi^2 R_\star^2 \frac{2c}{\lambda^4} kT \quad (19)$$

for the R-J limit. I'm making the approximation that we are in the R-J limit, although that isn't really true for the planetesimals. Nevertheless, the approximation will work well enough for this exercise. For the planetesimals the luminosity is

$$L_\lambda(\text{belt}) = N_p 4\pi R_p^2 \pi \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} = N_p 4\pi^2 R_p^2 \frac{2c}{\lambda^4} kT \quad (20)$$

where R_p is the radius of the planetesimals and N_P is the number of planetesimals given by $N_P = 3M_P/4\pi\rho_P R_P^3$ where ρ_P is the planetesimal density and M_P is the total mass of planetesimals. We can then write the luminosity of the belt as:

$$L_\lambda(\text{belt}) = \frac{3M_P}{4\pi\rho_P R_P^3} 4\pi^2 R_p^2 \frac{2c}{\lambda^4} kT = \frac{6c\pi M_P}{\rho_P R_P \lambda^4} kT \quad (21)$$

and the ratio is

$$\frac{L_\lambda(\text{belt})}{L_\lambda(\text{star})} = \frac{3M_P}{4\pi\rho_P R_P R_\star^2} \frac{T_P}{T_\star} \quad (22)$$

d. Assuming that the total combined mass of all the planetesimals was equal to the mass of the Earth, calculate the luminosity density of the belt as a fraction of the stellar luminosity density at the peak wavelength of the belt. Sketch the SED of the star plus the belt. The mass of the Earth is $6 \times 10^{27} \text{ gm}$. Assume the disk is very optically thin.

$$\frac{L_\lambda(\text{belt})}{L_\lambda(\text{star})} = \frac{3 \cdot 6 \times 10^{27} \text{ gm}}{4\pi \cdot 5 \text{ gm cm}^{-3} \cdot 10^5 \text{ cm} \cdot (7 \times 10^{10} \text{ cm}^2)} \left(\frac{273 \text{ K}}{6000 \text{ K}} \right) \quad (23)$$

for the M_P and R_P and ρ_P given above, the ratio is 0.027.

e. Now assume that the planetesimals were ground up into 1 cm diameter dust particles (perhaps by collisions), calculate the luminosity density of the belt as a fraction of the stellar

luminosity density at the peak wavelength of the belt. Sketch the SED of the star plus the belt. Assume the disk is still very optically thin.

Solution: in this case the ratio is 10^5 times larger, i.e. the belt is 2660 more luminous than the star at that wavelength. The reason is that you dramatically increased the surface area of the planetesimals by grinding them up into smaller pieces.

8. Using the SEDs below, identify one protostar (class I), one star with disk (class II), and one pre-main sequence star without disk (pure photosphere or class III). Identify stars by their phone number (73-???)

Solution: Class I is 213645. The Class III are 14-1229, 11-1659, and maybe 72-489. The remainder are Class II.