Solutions

1. *Calculate the L (luminosity) and T_{eff} for a deuterium burning star as a function of mass. Assume the star is convective with an opacity similar to a star on a Hayashi track, i.e. $\kappa_{rm} = \kappa_0 \rho T_{eff}^a$ where $a = 10$. First, write down the equations for L and T_{eff} for a convective star on a Hayashi track in terms of powers of R (radius) and M (mass) multiplied by an arbitrary constant (Lectures 18 or 22).*

The equations for a convective star with atmospheric opacity is $\kappa \propto \rho T^{10}$

$$T_{eff} \propto R^{1/5} M^{1/25} \tag{1}$$

$$L \propto R^{14/5} M^{4/25} \tag{2}$$

Next, convert from a dependence on R and M to a dependence on T_c (core temperature) and M.

$$T_c \propto M/R \tag{3}$$

or

$$R \propto M/T_c \tag{4}$$

Substituting R into Equations 1 & 2, we ge

$$T_{eff} \propto M^{6/25} T_c^{-1/5} \tag{5}$$

$$L \propto M^{74/25} T_c^{-14/5} \tag{6}$$

Then, consider a convective M-type star on the Hydrogen burning main sequence; the core temperature is $T_c = 10 \times 10^6$ K. A star on the main sequence with $M = 0.5\ M_\odot$ has a luminosity of 0.0455 $L_\odot$ and a $T_{eff} = 3800$ K; use this to calibrate the arbitrary constants.

$$T_{eff} = 3800\ K \left(\frac{M}{0.5\ M_\odot}\right)^{6/25} \left(\frac{T_c}{10^7 K}\right)^{-1/5} \tag{7}$$

$$L = 0.0455\ L_\odot \left(\frac{M}{0.5\ M_\odot}\right)^{74/25} \left(\frac{T_c}{10^7 K}\right)^{-14/5} \tag{8}$$

Finally, determine L and T_{eff} as a function of M for Deuterium burning, i.e. if the core temperature is kept at 1×10^6 K (using the constants you calibrated for the Hydrogen burning star). Sketch and compare the Hydrogen and Deuterium main sequences in the HR diagram for masses ranging from 0.1 to 0.5 $M_\odot$.

$$T_{eff} = 6000\ K \left(\frac{M}{0.5\ M_\odot}\right)^{6/25} \tag{9}$$

$$L = 29\ L_\odot \left(\frac{M}{0.5\ M_\odot}\right)^{74/25} \tag{10}$$

2.*Consider a 10 $M_\odot$ protostar with an intrinsic luminosity of 10^4 $L_\odot$ and a radius of 3 $R_\odot$. At what accretion rate does the total luminosity (intrinsic plus accretion luminosity) equal the Eddington luminosity (calculated for an ionized gas where Thompson scattering by electrons is the main source of opacity)?*

The Eddington Luminosity is given by:

$$L_{edd} = 3.2 \times 10^4 \left(\frac{M}{1 M_\odot}\right) L_\odot = 3.2 \times 10^5\ L_\odot \tag{11}$$

We then solve for

$$L_{edd} = L_{int} + L_{acc} = L_{int} + \frac{GM\dot{M}}{R} \tag{12}$$

or

$$\dot{M} = \frac{R}{G}(L_{edd} - L_{int}) = \frac{R}{GM}\left(3.2 \times 10^4 L_\odot \left(\frac{M}{M_\odot}\right) - L_{int}\right) \tag{13}$$

$$\dot{M}(M_\odot\ yr^{-1}) = \frac{3 \cdot 7 \times 10^{10}\ cm}{6.8 \times 10^{-8}\ cm^3\ gm\ s^{-1}} \left(\frac{4 \times 10^{33}\ erg\ s^{-1} \cdot \pi \times 10^7 s}{10 \cdot 2 \times 10^{33}\ gm}\right) (3.2 \times 10^5\ L_\odot - 10^4\ L_\odot) \tag{14}$$

which gives

$$\dot{M} = 0.003\ M_\odot\ yr^{-1} \tag{15}$$

How might a massive star still accrete once the luminosity exceeds the Eddington luminosity.

If the opacity is due to dust opacity, then if the gas and dust can be pushed to the dust sublimation radius by gas ram pressure, the dust will convert the radiation field to longer wavelengths where the opacity by dust is lower. Depending on the dust opacity law, this might lower the Rosseland mean opacity and allow the gas to accrete. However, the Thompson cross section used to calculate the Eddington luminosity is independent of wavelength. Another way to circumvent the Eddington limit is to use the flashlight effect. A disk may cause the luminosity to be preferentially beamed in certain directions. The resulting radiation field is non-isotropic. This could allow accretion in the directions in which the luminosity is lower, such as in the direction of the disk.

3. *What is the Kelvin-Helmoltz time? Which has a shorter Kelvin-Helmoltz time: a 10 or 1 $M_\odot$ pre-main sequence star? Why?*

The Kelvin Helmholtz time is the gravitational potential energy of a star over its luminosity:

$$t_{kh} = \frac{GM^2}{RL} \tag{16}$$

Since luminosity goes to the $\sim$ 4th power of M, the Kelvin Helmoltz time goes down with increasing mass.

4. *An infrared excess from a belt of planetesimals is detected with a peak wavelength of 10 μm around an A0V star with a luminosity of 40 $L_\odot$. How far is the belt from the star? Assume that the albedo of the dust grains is wavelength independent.*

The temperature of a star is determined by an equilibrium between emission and absorption. The energy absorbed will be:

$$\dot{E} = \frac{\pi a^2 Q L}{4\pi D^2} \tag{17}$$

$$\dot{E} = -4\pi a^2 Q \sigma T^4 \tag{18}$$

were a is the grain radius, Q is the emissivity, L is the luminosity of the star, D is the distance to the star and T is the temperature of the grain. Equating and solving:

$$T = \left(\frac{L}{16\pi\sigma}\right)^{1/4} D^{-1/2} \tag{19}$$

$$T = 288 \left(\frac{L}{L_\odot}\right)^{1/4} \left(\frac{D}{1AU}\right)^{-1/2} L_\odot \tag{20}$$

Thus if $\lambda T = 2900\ \ \mu$m K (Wiens law) then $T = 290$ K. The resulting distance is $D = 6$ AU.

5. *Calculate the L (luminosity), R (radius) and T_{eff} of a Helium burning star as a function of M (mass). First, calculate R as a function of T_c, μ and M. This equation will have powers of T_c, μ and M multiplied by a constant. Calibrate the constant by using a Hydrogen burning star. To do this, calculate μ for a solar abundance and use $T_c = 1.7 \times 10^7\ K$. This star should have a radius of 1 $R_\odot$ for a Mass of 1 $M_\odot$.*

$$P_c \propto \frac{M^2}{R^4} \tag{21}$$

Then using the ideal gas law, $P = \rho kT/\mu m_H$, then:

$$T_c \propto \frac{\mu M}{R} \tag{22}$$

or

$$R \propto \frac{\mu M}{T_c} \tag{23}$$

For solar abundances

$$\mu = \frac{4}{5 + 3x} \tag{24}$$

assuming X = 7.1 then $\mu = 0.6$. Thus

$$R = 1.7\ \left(\frac{M}{1\ M_{odot}}\right) \left(\frac{\mu}{0.6}\right) \left(\frac{10^7 K}{T_c}\right)\ R_\odot \tag{25}$$

Now, calculate μ for a pure Helium star and assume $T_c = 1. \times 10^8$ K. Using the calibrated constant, write R as a function M for the Helium star.

For a helium star, X=0, $\mu = 1.3$. Then

$$R = 0.374\ \ \left(\frac{M}{1\ M_\odot}\right)\ R_\odot \tag{26}$$

Next, calculate the luminosity. Start by assuming a radiative star where the energy transport is determined by the radiative diffusion equation:

$$L \propto \frac{RT^4}{\kappa\rho} \tag{27}$$

Assume a Kramers opacity ($\kappa \propto \rho T^{-3.5}$) that is independent of the composition of the star. Then, write the equation for L in terms of powers of M, T_c and μ multiplied by an constant. Again, to calibrate the constant, first calculate the constant for a Hydrogen burning main sequence star where the luminosity is transported through the star by radiation. Calculate μ for a solar abundance and use $T_c = 1.7 \times 10^7$ K. This star should have a luminosity of 1 $L_\odot$ for a Mass of 1 $M_\odot$.

$$L \propto \frac{RT^{6.5}}{\rho^2} \propto \frac{R^7 T^{7.5}}{M^2} \tag{28}$$

which if we substitute $R \propto \mu M / T_c$ then:

$$L \propto \frac{R^7 T^{6.5}}{M^2} \propto \mu^7 M^5 T_c^{0.5} \tag{29}$$

Calibrating

$$L = 0.77 \left(\frac{\mu}{0.6}\right)^7 \left(\frac{M}{1\ M_\odot}\right)^5 \left(\frac{T_c}{10^7\ K}\right)^{0.5}\ L_\odot \tag{30}$$

Next, calculate μ for a pure Helium star and assume $T_c = 1. \times 10^8$ K. What is the luminosity as a function of mass?

If $X = 0$ then $\mu = 1.33$. Then

$$L = 623 \left(\frac{M}{1\ M_\odot}\right)^5\ L_\odot \tag{31}$$

Finally, from R and L, calculate T_{eff} as a function of M for the Helium stars. Sketch and compare the Hydrogen and Helium main sequences in the HR diagram for masses of 1.0 to 2.5 $M_\odot$.

$$T_{eff} \propto \left(\frac{L}{R^2}\right)^{0.25} \propto \left(\mu^5 M^3 T_c^{2.5}\right)^{0.25} \propto \mu^{5/4} M^{1/2} T_c^{5/8} \tag{32}$$

$$T_{eff} = 4200 \left(\frac{\mu}{0.6}\right)^{5/4} \left(\frac{M}{1\ M_\odot}\right)^{3/4} \left(\frac{T_c}{10^7\ K}\right)^{5/8}\ K \tag{33}$$

Thus for a Helium star

$$T_{eff} = 42000 \left(\frac{M}{1\ M_\odot}\right)^{3/4}\ K \tag{34}$$

6. *Why does nuclear fusion maintain a fairly constant temperature in the core of a star? Why does the core of a star contract after fusion stops?*

Nuclear burning replaces the energy radiated into space. Due to the strong temperature dependence, a constant core temperature is maintained so that the nuclear burning matches the luminosity. If the rate or energy production is too high, the core heats up and then expands and cools until the energy production matches the luminosity. If the rate of energy production is too low, the core cools and then contracts and heats up until the energy production matches the luminosity.

The star begins to contract after fusion due to the fact that the core is no longer producing nuclear energy and it is self gravitating. The pressure for a given temperature is also reduced since μ has been doubled. The central core becomes self gravitating and is no longer just pressure confined. As the core radiates energy, it contracts, similar to a pre-main sequence star.

7. *Consider a gas surrounding a star. Calculate the net force from photon pressure and gravity on an optically thin gas as a function of distance from the star, the stellar luminosity, the stellar mass, the density of the gas and the opacity of the gas. If the force of photon pressure dominates over gravity near the star, does it dominate at large radii?*

The force of gravity on a parcel of gas with volume dV and density ρ is

$$F_{grav} = -\frac{GM\rho dV}{D^2} \tag{35}$$

where M is the mass of the star and D is the distance to the star. The force from photon pressure is

$$F_{rad} = \frac{\chi L \rho dV}{4\pi D^2 c^2} \tag{36}$$

where χ is the total opacity per mass. This gives a total pressure of:

$$F = F_{rad} - F_{grav} = \frac{\chi L \rho dV}{4\pi D^2 c^2} - \frac{GM\rho dV}{D^2} = \frac{\rho dV}{D^2}\left(\frac{\chi L}{4\pi c^2} - GM\right) \tag{37}$$

which is independent of distance. Thus, if radiation pressure dominates at small distances, since gravity and radiation pressure both decrease by the $1/distance^2$, radiation pressure will dominate at large distances.

8. *Look at the evolutionary tracks for a 1 $M_\odot$ star in Lecture 22 (the Iben 1967 plot). How hot will the Earth become during the peak luminosity of the red giant phase? Will this be hot enough to melt rock (900 K)? Assume the Earth's albedo is independent of wavelength.*

From problem 4:

$$T = 288\left(\frac{L}{L_\odot}\right)^{1/4}\left(\frac{D}{1\ AU}\right)^{-1/2} L_\odot \tag{38}$$

From the Iben plot, $log_{10}(L/L_\odot) = 2.5$

$$T = 288\left(\frac{10^{2.5}\ L_\odot}{L_\odot}\right)^{1/4}\left(\frac{1\ AU}{1\ AU}\right)^{-1/2} L_\odot = 1200\ K \tag{39}$$