Lecture 14: Viscous Accretion Disks

HST: 0.2 - 1 μm image of the A0e star AB Aurigae
Subgroups in the Orion OB1 Associations

The Orion OB1 association can be subdivided into four subgroups, each with a different age.
OB Associations as Laboratories for Disk Evolution

Hernandez et al. 2008
Orion OB1b

Squares and Circles: Members
Filled squares and circles: sources with disks

Spitzer image
OB Associations as Laboratories for Disk Evolution

Hernandez et al. 2008

OB1b: 5 Myr
OB Associations as Laboratories for Disk Evolution

Hernandez et al. 2008

OB1a: 10 Myr
OB Associations as Laboratories for Disk Evolution

Calvet et al. 2005
The formation of a disk

- Infalling matter collides with matter from the other side
- Forms a shock
- Free-fall kinetic energy is converted into heat

\[ \frac{kT}{\mu m_p} \approx \frac{1}{2} \frac{v_{ff}^2}{r} = \frac{GM_\star}{r} \]

At 10 AU from 1M\(_\odot\) star:
\[ T \approx 25000 \, K \]

- Heat is radiated away, matter cools, sediments to midplane
- Disk is formed
The formation of a disk

3-D Radiation-Hydro simulations of disk formation

Yorke, Bodenheimer & Laughlin 1993
Keplerian rotation (a reminder)

Disk material is almost (!) 100% supported against gravity by its rotation. Gas pressure plays only a minor role. Therefore it is a good approximation to say that the tangential velocity of the gas in the disk is:

\[ v_\phi \cong \Omega_K r = \sqrt{\frac{G M_*}{r}} \]

\[ \Omega_K \equiv \sqrt{\frac{G M_*}{r^3}} \]

Kepler frequency
The angular momentum problem

• Angular momentum of 1 $M_\odot$ in 10 AU disk:
  $3 \times 10^{53}$ cm$^2$/s
• Angular momentum of 1 $M_\odot$ in 1 $R_\odot$ star:
  $<<6 \times 10^{51}$ cm$^2$/s (=breakup-rotation-speed)
• Original angular momentum of disk = 50x higher than maximum allowed for a star
• Angular momentum is strictly conserved!
• Two possible solutions:
  – Torque against external medium (via magnetic fields?)
  – Very outer disk absorbs all angular momentum by moving outward, while rest moves inward: Need friction through viscosity!
Ring A moves faster than ring B. Friction between the two will try to slow down A and speed up B. This means: angular momentum is transferred from A to B.

Specific angular momentum for a Keplerian disk:

\[ l = r v_\phi = r^2 \Omega_K = \sqrt{GM_\ast r} \]

So if ring A loses angular momentum, but is forced to remain on a Kepler orbit, it must move inward! Ring B moves outward, unless it, too, has friction (with a ring C, which has friction with D, etc.).
Non-stationary (spreading) disks

Lynden-Bell & Pringle (1974), Hartmann et al. (1998)
Formation & viscous spreading of disk
Formation & viscous spreading of disk
Formation & viscous spreading of disk
Formation & viscous spreading of disk

From the rotating collapsing cloud model we know:

\[ r_{\text{centrif}} \sim t^4 \]

Initially the disk spreads faster than the centrifugal radius.

Later the centrifugal radius increases faster than disk spreading.
Formation & viscous spreading of disk

A numerical model

![Graph showing the distribution of mass density (Σ) with radius (R) over time (t = 6.3E+03 year).]
Formation & viscous spreading of disk

A numerical model
Formation & viscous spreading of disk

A numerical model

![Graph showing the evolution of surface density (Σ) vs. radius (R) with time (t = 6.3E+04 years).]
Formation & viscous spreading of disk

A numerical model
Formation & viscous spreading of disk

A numerical model
Viscosity

\[ \frac{dp_x}{dt} = l_{mf_p} v_{th} \rho \frac{\partial u_x}{\partial y} A \]  \hspace{1cm} (1)

where \( l_{mf_p} \) is the mean free path, which is equal to \( 1/(n\sigma) \) and \( v_{th} \) is the mean velocity from thermal motions.

\[ l_{mf_p} = \frac{1}{n\sigma} = \frac{\mu m_H}{\rho \sigma}, v_{th} = c_s = \sqrt{\frac{kT}{\mu m_H}} \]  \hspace{1cm} (2)

giving

\[ \frac{dp_x}{dt} = \frac{\mu m_H c_s}{\sigma} \frac{\partial u_x}{\partial y} A \]  \hspace{1cm} (3)
Viscosity

Now in a constant velocity shear, each layer receives as much momentum from the upper layer as it loses to the lower layers. Show a change in velocity, or a

\[
\frac{dp_x}{dt} = \frac{\partial}{\partial y} \left( \frac{\mu m_H c_s \frac{\partial u_x}{\partial y}}{\sigma} \right) A \Delta y
\]

we can divide out \(A \Delta y\) to determine the force unit per volume

\[
f_x = \frac{\partial}{\partial y} \left( \frac{\mu m_H c_s \frac{\partial u_x}{\partial y}}{\sigma} \right)
\]

this can be written as

\[
f_x = \frac{\partial \pi_{xy}}{\partial y}, \text{ where } \pi_{xy} = \rho \nu \frac{\partial u_x}{\partial y}
\]

\[
\nu = \frac{\mu m_H c_s}{\rho \sigma} = l_{mfp} c_s
\]
Shakura & Sunyaev model

Radial structure
Define the surface density:

\[ \Sigma(r) \equiv \int_{-\infty}^{+\infty} \rho(r,z) \, dz \]

Integrate continuity equation over z:

\[
\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial (\Sigma v_r)}{\partial r} = 0 \quad (1)
\]

Integrate radial momentum equation over z:

\[
\frac{\partial (\Sigma v_r)}{\partial t} + \frac{1}{r} \frac{\partial (\Sigma v_r^2)}{\partial r} + \frac{\partial (\Sigma c_s^2)}{\partial r} - \frac{v_\phi^2}{r} = -\frac{\Sigma GM}{r^2} \quad (2)
\]

Integrate tangential momentum equation over z:

\[
\frac{\partial (\Sigma l)}{\partial t} + \frac{1}{r} \frac{\partial (\Sigma rv_r l)}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left( \Sigma v r^3 \frac{\partial \Omega}{\partial r} \right) \quad (3)
\]
Shakura & Sunyaev model

Let’s first look closer at the radial momentum equation:

\[
\frac{\partial (\Sigma v_r)}{\partial t} + \frac{1}{r} \frac{\partial (\Sigma r v_r^2)}{\partial r} + \frac{\partial (\Sigma c_s^2)}{\partial r} - \Sigma \frac{v_\phi^2}{r} = -\Sigma \frac{GM}{r^2}
\]

Let us take the Ansatz (which one can later verify to be true) that \( v_r << c_s << v_\phi \).

\[
v_\phi^2 = \frac{GM}{r}
\]

That means: from the radial momentum equation follows the tangential velocity

\[
v_\phi = \sqrt{\frac{GM}{r}} \equiv \Omega_K r
\]

Conclusion: the disk is Keplerian
Let’s now look closer at the tangential momentum equation:

\[
\frac{\partial (\Sigma l_K)}{\partial t} + \frac{1}{r} \frac{\partial (\Sigma r v_r l_K)}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left( \Sigma v r^3 \frac{\partial \Omega_K}{\partial r} \right)
\]  

(3)

Now use continuity equation

\[
\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial (\Sigma r v_r)}{\partial r} = 0
\]

The derivatives of the Kepler frequency can be worked out:

\[
 v_r = - \frac{3}{\Sigma \sqrt{r}} \frac{\partial}{\partial r} \left( \Sigma v \sqrt{r} \right)
\]

That means: from the *tangential* momentum equation follows the *radial* velocity
Shakura & Sunyaev model

Radial structure

Our radial structure equations have now reduced to:

\[
\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial (\Sigma v_r)}{\partial r} = 0
\]

with

\[
v_r = - \frac{3}{\Sigma \sqrt{r}} \frac{\partial}{\partial r} (\Sigma v \sqrt{r})
\]

Missing piece: what is the value of \(\nu\)?

It is not really known what value \(\nu\) has. This depends on the details of the source of viscosity. But from dimensional analysis it must be something like:

\[
\nu = \alpha c_s h
\]

\[
\alpha = 0.001 \ldots 0.1
\]

Alpha-viscosity (Shakura & Sunyaev 1973)
Shakura & Sunyaev model

Further on alpha-viscosity:

\[ \nu = \alpha c_s h \]

Here the vertical structure comes back into the radial structure equations!

\[ h = \sqrt{\frac{kT r^3}{\mu m_p G M_*}} \equiv \frac{c_s}{\Omega_K} \]

So we obtain for the viscosity:

\[ \nu = \alpha \frac{c_s^2}{\Omega_K} \]
Shakura & Sunyaev model

Summary of radial structure equations:

\[
\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial (\Sigma r v_r)}{\partial r} = 0
\]

\[
v_r = -\frac{3}{\Sigma \sqrt{r}} \frac{\partial}{\partial r} \left( \Sigma v \sqrt{r} \right)
\]

\[
v = \alpha \frac{c_s^2}{\Omega_K}
\]

If we know the temperature everywhere, we can readily solve these equations (time-dependent or stationary, whatever we like).

If we don’t know the temperature a-priori, then we need to solve the above 3 equations simultaneously with energy equation.
Shakura & Sunyaev model

**Suppose we know that** $c_s^2$ **is a given power-law:**

Ansatz: surface density is also a powerlaw:

\[ c_s^2 \sim r^{-\xi} \]
\[ \Sigma \sim r^{-\eta} \]

The radial velocity then becomes:

\[ v_r \equiv -\frac{3}{\Sigma \sqrt{r}} \frac{\partial}{\partial r} \left( \Sigma \nu \sqrt{r} \right) = 3(\xi + \eta - 2) \frac{\nu}{r} \]

\[ v_r = -\frac{3}{2} \frac{\nu}{r} \]

Stationary continuity equation:

\[ \frac{\partial (\Sigma rv_r)}{\partial r} = 0 \]

from which follows:

\[ \Sigma \sim r^{\xi - 3/2} \]
\[ \eta = -\xi + 3/2 \]

Proportionality constants are straightforward from here on...
Shakura & Sunyaev model

Examples:

\[ T \sim r^{-3/2} \]

\[ \Sigma \sim r^{-0} \]
Shakura & Sunyaev model

Examples:

\[ T \sim r^{-1} \]

\[ \Sigma \sim r^{-1/2} \]
Shakura & Sunyaev model

**Examples:**

\[ T \sim r^{-1/2} \]

\[ \Sigma \sim r^{-1} \]
Shakura & Sunyaev model

Examples:

\[ T \sim r^{-0} \]

\[ \Sigma \sim r^{-3/2} \]
Shakura & Sunyaev model

Formulation in terms of accretion rate

Accretion rate $\dot{M}$ is amount of matter per second that moves radially inward through the disk.

$$\dot{M} = -2\pi r v_r \Sigma$$

Working this out with previous formulae:

$$v_r = -\frac{3}{2} \frac{v}{r}$$

$$\dot{M} = 3\pi \nu \Sigma$$

$$\dot{M} = 3\pi \alpha \frac{c_s^2}{\Omega_K} \Sigma$$

We finally obtain:

$$\Sigma = \frac{\dot{M}}{3\pi \alpha} \frac{\Omega_K \mu m_p}{k T_{\text{mid}}}$$

(but see later for more correct formula with inner BC satisfied)
Shakura & Sunyaev model

Effect of inner boundary condition:

Powerlaw does not go all the way to the star. At inner edge (for instance the stellar surface) there is an abrupt deviation from Keplerian rotation. This affects the structure of the disk out to many stellar radii:

\[
\Sigma = \frac{\dot{M}}{3\pi \alpha} \frac{\Omega_K \mu m_p}{k T_{\text{mid}}} \left[ 1 - \left( \frac{r_{\text{in}}}{r} \right)^{1/2} \right]
\]

Keep this in mind when applying the theory!
Shakura & Sunyaev model

How do we determine the temperature?

We must go back to the vertical structure......and the energy equation.....

First the energy equation: heat production through friction:

\[ Q_+ \equiv \Sigma v \left( r \frac{\partial \Omega_K}{\partial r} \right)^2 = \frac{9}{4} \Sigma v \frac{G M_*}{r^3} \]

For power-law solution: use equation of previous page:

\[ v_r = -\frac{3}{2} \frac{v}{r} \quad \rightarrow \quad v = -\frac{2}{3} r v_r \]

The viscous heat production becomes:

\[ Q_+ = -\frac{3}{4 \pi} 2 \pi \Sigma v_r \frac{G M_*}{r^3} \]
Shakura & Sunyaev model

How do we determine the temperature?

\[ Q_+ = -\frac{3}{4\pi} 2\pi \Sigma r v_r \frac{GM_*}{r^3} \]

Define ‘accretion rate’ (amount of matter flowing through the disk per second):

\[ \dot{M} \equiv -2\pi \Sigma r v_r = \text{constant} \]

End-result for the viscous heat production:

\[ Q_+ = \frac{3}{4\pi} \dot{M} \Omega_K^2 \]
Shakura & Sunyaev model

How do we determine the temperature?

Now, this heat must be radiated away.

Disk has two sides, each assumed to radiate as black body:

\[ 2\sigma T_{\text{eff}}^4 = Q_+ = \frac{3}{4\pi} \dot{M} \Omega_K^2 \]

One obtains:

\[ T_{\text{eff}} = \left( \frac{3}{8\pi\sigma} \frac{\dot{M} \Omega_K^2}{\dot{M} \Omega_K^2} \right)^{1/4} \sim r^{-3/4} \]
Molecular viscosity? No!

Problem: molecular viscosity is virtually zero

Reynolds number

\[ \text{Re} = \frac{<u> L}{\nu} \]

\( <u> \) = typical velocity
\( \nu \) = viscosity
\( L \) = length scale

Molecular viscosity:

\[ \nu = <u_T> l_{\text{free}} \]

\( l_{\text{free}} \) = m.f.p. of molecule
\( <u_T> \) = velo of molecule

Typical disk (at 1 AU):

\( \text{N}=1 \times 10^{14} \text{ cm}^{-3}, \text{T}=500 \text{ K}, \text{L}=0.01\text{AU} \)

Assume (extremely simplified) \( \sigma_{\text{H}_2} \approx \pi (1\text{Ang})^2 \).

\[ <u_T> = \sqrt{\frac{3kT}{\mu m_p}} = 2.3 \text{ km/s} \]

\[ \nu = 7.3 \times 10^6 \text{ cm}^2/\text{s} \]

\( l_{\text{free}} = \frac{1}{N \sigma} = 32 \text{ cm} \)

\( \text{Re} = 4.7 \times 10^9 \)
Magneto-rotational instability (MRI)
(Also often called Balbus-Hawley instability)

Highly simplified pictographic explanation:

If a (weak) pull exists between two gas-parcels A and B on adjacent orbits, the effect is that A moves inward and B moves outward: a pull causes them to move apart!

The lower orbit of A causes an increase in its velocity, while B decelerates. This enhances their velocity difference! This is positive feedback: an instability.

Causes turbulence in the disk
Magneto-rotational instability (MRI)

Johansen & Klahr (2005); Brandenburg et al.
Gravitational (in)stability

If disk surface density exceeds a certain limit, then disk becomes gravitationally unstable.

Toomre Q-parameter:

\[ Q = \frac{h \Omega_k^2}{\pi G \Sigma} \approx \frac{h}{r} \frac{M_*}{M_{\text{disk}}} \]

For \( Q > 2 \) the disk is stable
For \( Q < 2 \) the disk is gravitationally unstable

Unstable disk: spiral waves, angular momentum transport, strong accretion!!
Gravitational (in)stability

Spiral waves act as `viscosity'

Rice & Armitage
FU Orionis stars
McNeal’s Nebula: a new FU Ori?
The Luminosity Problem

This is a longstanding problem: the required accretion luminosities are on average higher than the observed luminosities.

Grey gives predicted luminosities from models.
Summary

• In Keplerian orbits, the inner disk is rotating faster than the outer disk.
• Viscosity acts as a friction. Mass is transported inward, angular momentum outward.
• This causes mass to accrete onto the central source (Will Fischer’s lecture Tuesday), and the disk to spread outward.
• Viscosity can be due to thermal motions in a gas (not important for disks), magneto rotational instability, or spiral waves created by gravity.
• Young stars may go through episodic accretion: gas first accumulates onto a disk until it undergoes gravitational instability, then the mass is dumped onto the disk in a major event.